ENSEMBLE AVERAGES of WIGNER DISTRIBUTIONS of NOISE and TELECOMMUNICATION SIGNALS with EMPHASIS on the ROLE of CROSS-TERMS

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Abstract: The paper presents a study of properties of ensemble averages of Wigner time-frequency distributions (WDs) of random processes defined by statistically independent samples of stationary and nonstationary Gaussian noise and of radio-frequency telecommunication signals. We used samples of PSK and FSK signals transmitting random telegraph signals. The WDs of single samples of these random processes have the form of random bipolar fields $W(t, f)$, while ensemble averages $E\{W(t, f)\}$ are well defined deterministic functions. The WDs of real signals have the form of a sum of an even term and a cross-term. In selected cases, theoretical forms of ensemble averages of these terms are derived and compared with computer simulations. In other cases, only computer simulations are applied. It was shown that ensemble averages of even terms are usually (but not always) unipolar and of cross-terms bipolar well-defined deterministic functions. For so-called proper processes the ensemble averages of cross-terms are cancelled. In computer simulations, their level decreases with increasing number of samples. In other cases, cross-terms carry an information about some properties of a random process. Derivations show that the notion of a cross-term coincides with the double value of the real part of the so-called complementary Wigner distribution. In consequence, the paper yields the answer to the question, in which cases the complementary distribution matters and yields a deeper insight into the properties and role of cross-terms.

Keywords – Wigner distributions, random analytic signals and processes. ensemble averages.

EDICS: SSP-SSAN, SSP-SPEC, SSP-SNMD, SPC-APPL.
1. INTRODUCTION

The Wigner distribution (WD) [1] is the most popular member of time-frequency distributions. Its theoretical background and applications are described in numerous papers and summarized in many books, for example [2], and handbooks, for example [3]. In the case of random processes, we have to distinguish WDs of single sample functions (realizations) $W(t, f)$ and its ensemble averages $E\{W(t, f)\}$. Their properties are different. The WD of a single sample of a random process, for example a sample of Gaussian noise, has the form of a bipolar random field. Differently, the ensemble average $E\{W(t, f)\}$ is a well defined deterministic function. Generalized expressions in this context are given in [4]. The WD of a real signal contains so-called cross-terms (see Eq.1 in Section 2). The authors of [5] describe the role of ensemble averages of a so-called complementary WDs of analytic signals. However, the notion of a cross-term coincides with the double real part of the complementary WD. In this paper, in Section V, for selected random processes estimates of ensemble averages of cross-terms are derived or calculated yielding a deeper insight into its properties.

Let us have a comment about computer calculations of ensemble averages. Theoretically, the operator $E\{\}$ requires the summation of infinite number of samples. In computer simulations, the number of samples is finite. In this paper, to get a reasonable accuracy we used 200 to 2000 statistically independent samples. It is hard to imagine that such amounts could be available from experiments. We should apply a theoretical model of generation samples of random signals. If the construction of such a model is impossible, the ensemble averages cannot be calculated. This paper is illustrated with estimates of ensemble averages of WDs of noise and telecommunication signals calculated using theoretical models. Let us mention that if the correlation function of the random process can be derived in a closed form, the ensemble average of the WD defined by the Fourier transform of the correlation function may also have a closed form. Differently, the WD of a single sample, for example a sample of a low-pass Gaussian noise, usually cannot be derived in a closed form. It can be computed if the data defining the sample are available. In that case, the knowledge of the closed form validates the numerical calculation of the ensemble average of the WD.

II. NOTATIONS of WIGNER DISTRIBUTIONS

The Wigner distribution (WD) [1] of a real signal $x(t)$ may be written in the form [6]

$$ W_{ss}(t,f) = 0.5W_{even} + 0.25W_{cross} = 0.25\left[ W_{ss}(t,f) + W_{ss}(t,-f) + W_{cross}(t,f) \right], \quad (1) $$

where $W_{ss}(t,f)$ is the WD of the analytic signal $s(t) = x(t) + j\tilde{x}(t)$. Here $\tilde{x}(t)$ is the Hilbert transform of $x(t)$ and $W_{ss}(t,-f) = W_{\tilde{x}s}(t,f)$ is the WD of the conjugate analytic signal $s^*(t)$. The term $W_{even}(t,f) = 0.5[ W_{ss}(t,f) + W_{ss}(t,-f) ]$ is called the even part. Let us remind that due to the bilinear nature, the WD of a real signal contains a cross term $W_{cross}(t,f)$. Boashash [7] proposed to apply $W_{ss}(t,f)$ instead of $W_{ss}(t,f)$ to avoid the generation of $W_{cross}(t,f)$. However, the term $W_{even}(t,f)$ does also not contain the cross-term at the cost of containing the redundant term $W_{ss}(t,-f)$. The cross-term $W_{cross}(t,f) = W_{cross}(t,-f)$ is also an even function of $f$. Therefore, all terms with the support in the half-plane $f < 0$ are redundant. In this paper, to get a more illustrative presentation, we decided to display in examples all three terms of (1). Note that it is easier to compute $W_{ss}$ than $W_{ss'}$. Some important features of the cross-terms are studied by slices of the 2-D time-frequency distributions along the line $f = 0$. Such a slice requires a two-sided representation in the frequency domain. For low-pass signals, the supports of $W_{even}$ and $W_{cross}$ overlap and both terms should be displayed separately. In that case, we may calculate the WD of the Hilbert transform $\tilde{x}(t)$ of the form

$$ W_{ss}(t,f) = 0.5W_{even} - 0.25W_{cross}. \quad (2) $$

It differs from (1) only by the sign of $W_{cross}(t,f)$. The addition of (1) and (2) yields

$$ W_{even} = W_{xx}(t,f) + W_{\tilde{x}s}(t,f) \quad (3) $$

and the subtraction yields

$$ W_{cross}(t,f) = 2\left[ W_{xx}(t,f) - W_{\tilde{x}s}(t,f) \right]. \quad (4) $$
III. CORRELATION PRODUCTS AND FUNCTIONS

Let us define the correlation product for a single sample $s_i(t)$ of a random process $\{s(t)\}$

$$
\hat{r}_{ax}^{(i)}(t, \tau) = s_i(t + 0.5\tau)s_i^*(t - 0.5\tau) = x_i^*x_i^- + \bar{x}_i^\tau \bar{x}_i^- + j\left(x_i^*\bar{x}_i^- - \bar{x}_i^\tau x_i^-\right)
$$

(5)

and the complementary correlation product [5]

$$
\hat{r}_{ax}^{(i)}(t, \tau) = s_i(t + 0.5\tau)s_i(t - 0.5\tau) = x_i^*x_i^- - \bar{x}_i^\tau \bar{x}_i^- + j\left(x_i^*\bar{x}_i^- + \bar{x}_i^\tau x_i^-\right),
$$

(6)

where the superscript “+” denotes a function of $t + 0.5\tau$ and “-” a function of $t - 0.5\tau$. Let us denote the terms of (5) as follows

$$
\hat{r}_{ax}^{(i)}(t, \tau) = r_{ax}^{(i)} + j\left(r_{ax}^{(i)} - r_{ax}^{(-i)}\right).
$$

(7)

Therefore, the terms of the complementary product are

$$
\hat{r}_{ax}^{(i)}(t, \tau) = r_{ax}^{(i)} - r_{ax}^{(-i)} + j\left(r_{ax}^{(i)} + r_{ax}^{(-i)}\right).
$$

(8)

The ensemble averages of correlation products define the corresponding correlation functions. We have

$$
\hat{r}_{ax}^{(i)}(t, \tau) = E\{\hat{r}_{ax}^{(i)}(t, \tau)\} = r_{ax}^{(i)} + r_{ax}^{(-i)} + j\left[r_{ax}^{(i)} - r_{ax}^{(-i)}\right]
$$

(9)

and the complementary correlation function

$$
\hat{r}_{ax}(t, \tau) = E\{\hat{r}_{ax}^{(i)}(t, \tau)\} = r_{ax}^{(i)} - r_{ax}^{(-i)} + j\left[r_{ax}^{(i)} + r_{ax}^{(-i)}\right],
$$

(10)

where $r_{ax}$ and $r_{ax}$ are autocorrelation and $r_{ax}$ and $r_{ax}$ cross-correlation functions.

Proper and improper processes: The authors of [5] following the ideas presented by Picinbono et all. (see reference [3]-[6] in [5]) define so-called proper and improper complex random signals. A complex zero-mean random signal $s(t)$ is called proper if $E\{s(t_1)s(t_2)\} = 0$ for all pairs $(t_1, t_2)$. Here we have $t_1 = t + 0.5\tau$ and $t_2 = t - 0.5\tau$. The signal is proper if in (5) $r_{ax} = r_{ax}$ and $r_{ax} = -r_{ax}$. Important remark: For random signals, the correlation products defined by (5) and (6) are random functions. Differently, the ensemble averages (9) and (10) are well-defined deterministic functions [4]. Note that for proper processes, the real and imaginary terms of the complementary correlation function equal zero.
IV ENSEMBLE AVERAGES OF WIGNER DISTRIBUTIONS

The WD of a single sample of the analytic signal $s_i(t)$ defined by the Fourier transform of the correlation product (5)

$$W_{si}^{(i)}(t, f) = \int_{-\infty}^{\infty} r_{si}^{(i)}(t, \tau) e^{-j2\pi f \tau} d\tau$$

(11)

is a real function. The same formula applies for the WD of a real signal $x(t)$ changing the subscript $ss^*$ to $xx$. Differently, the complementary WD defined by the Fourier transform of the complementary correlation product (6), i.e.,

$$W_{si}^{(i)}(t, f) = \int_{-\infty}^{\infty} r_{si}^{(i)}(t, \tau) e^{-j2\pi f \tau} d\tau$$

(12)

is a complex function. The insertion of (6) disregarding two vanishing integrals yields

$$W_{si}^{(i)}(t, f) = \int_{-\infty}^{\infty} r_{si}^{(i)}(t, \tau) \cos(2\pi f \tau) d\tau + j \int_{-\infty}^{\infty} r_{si}^{(i)}(t, \tau) \cos(2\pi f \tau) d\tau .$$

$$W_{si}^{(i)}(t, f) = \text{Re}[W_{si}^{(i)}(t, f)] + j \text{Im}[W_{si}^{(i)}(t, f)].$$

(13)

The WD of a single sample of a random signal defined by (11) may be regarded as sample of a random proces \{W_{si}^{(i)}(t, f)\}. The ensemble average is

$$T_{si}^{(i)}(t, f) = E\{W_{si}^{(i)}(t, f)\} = E\left\{\int_{-\infty}^{\infty} r_{si}^{(i)}(t, \tau) e^{-j2\pi f \tau} d\tau\right\} .$$

(14)

We apply the notation $T$ used in [5]. Since the operator $E$ and the Fourier transform are linear, the order in (14) can be changed. We get

$$T_{si}^{(i)}(t, f) = E\{W_{si}^{(i)}(t, f)\} = \int_{-\infty}^{\infty} E\{r_{si}^{(i)}(t, \tau)\} e^{-j2\pi f \tau} d\tau = \int_{-\infty}^{\infty} r_{si}^{(i)}(t, \tau) e^{-j2\pi f \tau} d\tau .$$

(15)

This form shows that the ensemble average $T_{si}^{(i)}(t, f)$ is given by the Fourier transform of the correlation function defined by (9) (see [2], Section 3.2.5). In consequence, it is also a well-defined deterministic function [4]. Computer simulations with a finite number of $N$ samples confirmed that the Eq.(14) and (15) yield the same functions. Note that (15) requires the summation of $N$ correlation products and calculation of a single integral while (14) requires the calculation of $N$ integrals. Obviously, the computational efficiency is much better by implementation of (15) than (14).

Similarly to (15), the ensemble average of the complementary WD is
\[ T_w(t, f) = \int_{-\infty}^{\infty} r_w(t, \tau) e^{-j2\pi f \tau} d\tau = \text{Re}\{T_w\} + j\text{Im}\{T_w\} \quad \text{(16)} \]

It can be shown that the cross-term in (1) is equal to twice the real part of the complementary WD defined by (13), i.e., \( T_{\text{cross}}(t, f) = 2\text{Re}\{T_w(t, f)\} \). Remark: In [5] the last term in Eq.(28), i.e., \( \text{Re}T^W_w(t, f) \) should be multiplied by 2. Concluding, investigations about the role of the complementary WD coincide with investigations about the role of cross-terms and for proper processes the theoretically derived ensemble averages of cross-terms equal zero.

**The nature of the terms of the ensemble average of the WD of a real signal.**

The WD of a single sample of a real signal \( x_i(t) \) is given by the Eq.(11). The corresponding ensemble average is (see (1))

\[ T_{\text{xx}}(t, f) = 0.5T_{\text{even}} + 0.25T_{\text{cross}} = 0.25[T_{x'i'}(t, f) + T_{x'i}(t, f) + T_{\text{cross}}(t, f)] \quad \text{(17)} \]

and of the Hilbert transform \( \tilde{x}_i(t) \) (see (2))

\[ T_{\tilde{x}\tilde{x}}(t, f) = 0.5T_{\text{even}} - 0.25T_{\text{cross}} = 0.25[T_{x'i'}(t, f) + T_{x'i}(t, f) - T_{\text{cross}}(t, f)] \quad \text{(18)} \]

The energies of the signal and its Hilbert transform are equal and given by the integrals [3]

\[ \text{Energy} = \int \left[ x(t) \right]^2 dt = \int \left[ \tilde{x}(t) \right]^2 dt = \int\int T_{\text{xx}}(t, f) dt df = \int\int T_{\tilde{x}\tilde{x}}(t, f) dt df. \quad \text{(19)} \]

This equation is fulfilled only if the energy of the cross term in (17) and (18) equals zero. We have

\[ \int\int T_{\text{cross}}(t, f) dt df = 0. \quad \text{(20)} \]

In many presented examples this equation is fulfilled due to the periodicity in time of \( T_{\text{cross}}(t, f) \). Differently, the two terms of \( T_{\text{even}} \) are usually (but not always) unipolar functions. Both are well defined deterministic functions. Note that (19) and (20) are valid if we insert \( W_{xx} \) in place of \( T_{xx} \) [6].

**V. EXAMPLES**

**V.I. Noise signals**

1. **Low-pass noise**

Consider samples \( x_i(t) \) of a low-pass nonstationary Gaussian noise with the power density
If \( B(t) = B \) is a constant, the noise is WSS. Fig. 1a shows a sample of \( x_i(t) \), \( B = 1 \), generated by the method described in [8], [9]. The method enables the generation of the Hilbert transform of \( x(t) \). A sample of the nonstationary noise is shown in Fig. 1b. The corresponding instantaneous bandwidth

\[
B(t) = 1 + \left[ 0.25 + 0.25 \tanh \left( 0.16(t - 40) \right) \right]
\]

is shown in Fig. 1c.

**The stationary case**

In the stationary case, the terms of the correlation function (9) are time independent and have the form [9]

\[
r_{xx}(\tau) = r_{xx}(\tau) = \int_{-B}^{B} G_0 e^{-j2\pi f \tau} df = 2G_0 B \frac{\sin(2\pi B \tau)}{2\pi B \tau},
\]

\[
r_{xx} = -r_{xx} = 2G_0 B \frac{1 - \cos(2\pi B \tau)}{2\pi B \tau}.
\]

The random process \( \{x_i(t)\} \) is WSS and proper and the terms of the complementary correlation function (10) equal zero. Therefore, the ensemble average of cross-terms \( T_{cross} \) equals zero. In consequence, the ensemble average of the WD given by (17) is \( T_{xx}(t, f) = 0.5T_{even} = G(f) \otimes \delta_1 \). The multiplication with \( \delta_1 \) indicates the time independence. Let us explain why having the theoretical solution it is still reasonable to calculate computer simulations.

1. The samples of noise of Fig. 1 are generated by a computer using a specific algorithm. They represent a simulation of the ideal Gaussian noise. Simulations are never perfect.
2. The theoretical ensemble average is defined for an infinite number of samples. In computer simulations this number is finite. In consequence, the cancellation of the cross-term is not perfect. In the example shown below, the vanishing cross-term is modulated by a sine wave. Actually, we have not derived a theoretical explanation of this effect.

3. There are no closed forms representing the WDs of single samples. Therefore, comparisons of the properties of WDs of single samples with the ensemble averages are possible using computer simulations.

4. Computer simulations of examples with known theoretical solutions validate similar simulations for which theoretical solutions are not derived.

5. Note that computer simulations apply samples of finite length while the theoretical samples may have infinite length.

Let us present the result of computer simulations for the low-pass stationary noise. The WD of a single sample \( x_i(t) \) defined by Eq.(1) is displayed in Fig.2. Since the supports of the two terms of \( W_{xx}(t,f) = 0.5W_{\text{even}} + 0.25W_{\text{cross}} \) overlap, we display: a) \( W_{xx} \), b) \( 0.5W_{\text{even}} \) and c) \( 0.25W_{\text{cross}} \). We observe that all terms defined by the Eq.(1) are bipolar random fields. The corresponding estimates of ensemble averages calculated using 2000 samples are shown in Fig.3 and 4. We observe that all terms of (17) are well defined deterministic functions and the random process \( \{ x_i(t) \} \) is WSS and proper. From Fig.4b and c we can see, that \( T_{\text{cross}}(t,f) \) is a bipolar function due to the modulation by a sine wave of frequency \( f_{\text{cross}} = 2B \). Let us remind that the zero crossing frequency of a single sample of noise equals \( 2B/\sqrt{3} \) (Rice formula [10]). However, the comparison of Figs.2c and 3c shows that the averaging process, as expected, almost cancels the cross-term. The slice \( T_{\text{even}}(f,t=20) \) displayed in Fig.4a (solid line) shows that the term \( T_{\text{even}}(t,f) \) is also a well defined deterministic function. This slice is a good approximation of the power spectrum given by Eq.(21). The dotted line represents the vanishing cross-term. Theoretically, the power spectrum of Fig.4a should be a rectangular function.
Fig. 2. The WDs of a single sample of noise (see Fig. 1a). a) $W_{xx}(t,f) = 0.5W_{\text{even}} + 0.25W_{\text{cross}}$, b) $0.5W_{\text{even}}$, and c) $0.25W_{\text{cross}}$. In this example the supports of $W_{\text{even}}$ and of $W_{\text{cross}}$ overlap.

Fig. 3. Estimates of ensemble averages for 2000 samples corresponding to Fig. 2: a) $T_{xx}(t,f) = 0.5T_{\text{even}} + 0.25T_{\text{cross}}$, b) $T_{\text{even}}$, and c) The vanishing term $T_{\text{cross}}$.

Fig. 4. a) Slices of $T_{xx}(t, f=20)$. Solid line $T_{\text{even}}$, dotted line $T_{\text{cross}}$. b) The slice $T_{\text{cross}} (t,f=0)$ shows the sine wave modulating the envelope of $T_{\text{cross}}$. c) A fragment of the $t$-axis of b) shows a sine wave.

The nonstationary case

Fig. 5. Nonstationary low-pass noise. The estimates of ensemble averages. a) $T_{xx}(t,f) = 0.5T_{\text{even}} + 0.25T_{\text{cross}}$, b) $0.5T_{\text{even}}$, c) $0.25T_{\text{cross}}$.

For the nonstationary noise with a sample function of Fig. 1b, we display only the estimates of the ensemble averages calculated using 2000 samples. Fig. 5 shows these averages in the same order as in
Fig.3. The cross-term in Fig.5c is negligible. A computer simulation cannot serve as an exact evidence. Nevertheless, it shows that the presented nonstationary process is proper. Fig.6 shows that the vanishing cross-term is modulated by a sine wave. Detailed observations (not displayed) show that the frequency of this modulation decreases in time by about 3.8% while the bandwidth increases by 50%. The change of the badwidth is illustrated by the slices of $T_{\text{even}}$ displayed in Fig.7.

![Graphs showing the nonstationary Gaussian noise](image)

Fig.6. Nonstationary Gaussian noise. The slices of the terms of $T_{\alpha}(t, f = 0)$. a) Solid line: $T_{\text{even}}(t, f = 0)$, dotted line $T_{\text{cross}}(t, f = 0)$ 2, b) A fragment for $18 < t < 22$ of $5T_{\text{cross}}(t, f = 0)$.

![Graphs showing the nonstationary Gaussian noise](image)

Fig.7. Nonstationary Gaussian noise. The slices $T_{\alpha}(t = \text{const}, f)$. a) $t = 1.25$, b) $t = 19.85$ and c) $t = 35.0$. Solid lines: $T_{\alpha}(t = \text{const}, f)$ show the change of the bandwidth in time, dotted lines: negligible values of $T_{\text{cross}}(t = \text{const}, f)$.

2. Band-pass Gaussian noise.

The power spectrum of a band-pass Gaussian noise can be regarded as the difference of two low-pass power spectra given by (21) with $B_2 > B_1$. The corresponding time independent autocorrelation functions are

$$r_{xx}(\tau) = r_{\alpha \alpha}(\tau) = 2G_\alpha B_2 \frac{\sin(2\pi B_2 \tau)}{2\pi B_2 \tau} - 2G_\alpha B_1 \frac{\sin(2\pi B_1 \tau)}{2\pi B_1 \tau}$$

(23)

Using (22), we get the cross-correlation functions. Of course, the band-pass Gaussian noise is proper and, as in the case of a low-pass noise, the cross-term vanish. The results of computer simulations are presented in Fig.8 and 9. In this example the supports of $T_{\text{even}}$ and $T_{\text{cross}}$ are disjoint.
Fig. 8. Band-pass noise, $B_2 = 2$, $B_1 = 1$. a) The ensemble average $T_{xx}(t, f) = 0.5T_{\text{avg}} + 0.25T_{\text{cross}}$. b) The vanishing term $T_{\text{cross}}$.

Fig. 9. Band-pass noise, the slices of the WD of Fig. 8. a) $T_{xx}(t = 20, f)$. The solid line shows the estimate of the power spectrum and the dotted line the vanishing cross-term. b) A part of the t-axis of the slice $T_{xx}(t, f = 0)$, average for 2000 samples (note the range of the y-axis). c) Slices of calculated correlation functions $r_{xx}(t, \tau)$ and $r_{xy}(t, \tau)$. Solid line $r_{xx}(t = 20, \tau)$ and dotted line $r_{xy}(t = 20, \tau)$. The difference is almost invisible.

The slices of calculated real parts of the correlation function defined by Eq. 9, displayed in Fig. c, confirm that the $r_{xx} r_{xy}$, i.e., the Gaussian band-pass noise is proper. The theoretical shape of these functions is given by the Eq. (23).

3. Modulated low-pass noise

A sample function of this process has the form

$$s_i(t) = y_i(t)e^{i2\pi f_c t} = y_i(t)\cos(2\pi f_c t) + jy_i(t)\sin(2\pi f_c t) = x_i(t) + j\tilde{x}_i(t),$$

where $y_i(t)$ is a sample function of the low-pass WSS noise of power spectrum defined by (21).

Note that the imaginary part is the Hilbert transform of the real part only, if $f_c \geq B$, i.e., the Bedrosian’s theorem can be applied [11]. The correlation functions are

$$r_{xx}(t, \tau) = \rho_{xx}(\tau)[\cos(2\pi f_c \tau) + \cos(4\pi f_c t)],$$

$$r_{xy}(t, \tau) = \rho_{xy}(\tau)[\cos(2\pi f_c \tau) - \cos(4\pi f_c t)].$$
where \( \rho_{yy}(\tau) \) has the form (22). Evidently, \( r_{xx}(t,\tau) \neq r_{xx}(t,\tau) \), i.e., the process is improper. The calculation of the ensemble averages of WDs yields

\[
T_{xx}(t,f) = \int 0.5 \rho_{yy}(\tau) \left[ \cos(2\pi f_c \tau) + \cos 4\pi f_c \tau, t \right] e^{-j2\pi f_c \tau} d\tau
\]

\[
= 0.25 [G(f - f_c) + 0.25G(f + f_c)] \otimes 1, + 0.5G(f) \cos(4\pi f_c \tau)
\]

(27)

\[
T_{xx}(t,f) = 0.25 [G(f - f_c) + 0.25G(f + f_c)] \otimes 1, - 0.5G(f) \cos(4\pi f_c \tau)
\]

(28)

Note that (27) is a specific case of (17) and (28) of (18). Therefore, the first two terms in (27) and (28) represent the unipolar term \( T_{even} \) and the third one the bipolar term \( T_{cross} \). Fig.10 shows these terms calculated using the WSS noise of Fig.1a, \( B = 0.5 \) and a carrier with \( f_c = 1.5 \). Here, the three terms of (17) have disjoint supports. The comparison of Figs.8 and 9 with Fig.10 shows, that differently to the band-pass noise, the term \( T_{cross} \) does not vanish and is modulated by a sine wave with \( f_{cross} = 2f_c = 3.0 \), i.e., the same as the frequency of the cross term of the WD of a real signal \( \cos(2\pi f_c \tau) \) (carrier), i.e., \( \cos(4\pi f_c \tau) \).

A comment to the interpretation of an example by the authors of [5]

The authors of [5] defined a random process with sample functions given by the real part of (24), however, with the carrier frequency \( f_c = f_{c,i} \) defined as a random variable. The example presented in [5] applies \( f_{c,1} = B \) and \( f_{c,2} = 2B \), where \( B \) is the bandwidth of a WSS low-pass noise with a power spectrum given by (21). (Remark: In [5], the bandwidth defined in the text differs from the bandwidth displayed in Fig.3). Let us quote the conclusion of the authors of [5]: "Thus, \( T_{wx}^w(t,f) = \)
\[ \sum I_{\gamma}(f - f_{c,0}) \] is constant in \( t \), whereas only \( T_{\gamma}(t, f) = \sum I_{\gamma}(f) \exp(j2\pi \cdot 2nf_{c,0}t) \) displays a periodic behaviour in \( t \). Therefore, a description based on \( T_{\gamma}(t, f) \) fails to capture the cyclo-stationary nature of \( x(t) \). Rather, it would lead us to wrongly conclude that \( x(t) \) is WSS” – end of citation. Our comment is as follows: Having in mind the equiprobable values of the two carrier frequencies, the process defined in [5] may be regarded as a sum of two processes defined by (24) weighted by 0.5. In consequence, the resulting \( T_{xx} \) is a sum of two \( T_{xx} \) given by (27). The computer simulation for \( B = 0.5 \) is shown in Fig.11. The terms \( T_{\text{even}} \) overlap and the cross term \( T_{\text{cross}} \) has the envelope \( G(f) \) modulated by the waveform of Fig.11c. Instead of the sine wave of Fig.10c, we observe a sine wave distorted by the second harmonic. The authors of [5] classify this process as cyclostationary defined by the periodicity of the cross-term w.r.t. the variable \( t \). However, usually the process is called cyclostationary if the correlation function \( r(t, \tau) \) is periodic w.r.t. the shift variable \( \tau \). Note that using the definition of cyclostationarity proposed in [5], the process defined by sample functions (24) should also be classified as cyclostationary. We are not in favour to use such a definition.

![Fig.11.Illustration of the example presented in [5]. Computer simulation of \( T_{xx} \) using \( B = 0.5, f_{c,1} = B \) and \( f_{c,2} = 2B \). a) \( T_{xx}(t, f) \), b) The slice \( T_{xx}(t = 20.5, f) \), solid line \( T_{\text{even}} \), dotted line \( T_{\text{cross}} \). c) The periodic waveform of the slice \( T_{\text{cross}}(t, f = 0) \).](image)

### V.2. Telecommunication Signals

All WDs of single samples of random telecommunication signals have the form of random bipolar fields similar as in Fig.2. Examples are here not presented [12], [13], except the case of FSK.

#### 1. Phase-Shift Keying Signals

A sample of a radio frequency analytic signal with phase shift keying (PSK) is given by the formula

\[ s_r(t) = A_0 e^{j[2\pi f_t t + \phi(t + \tau)]} \]  \quad (29)
We assume that $b_i(t)$ is a sample of a binary random telegraph process shown in Fig.12a. It was generated using a method described in [12]. The signal is analytic for carrier frequencies sufficiently high (no leakage of the spectrum into negative frequencies [11]).

**Synchroneous carrier versus asynchroneous carrier**

The phase constant $\theta_i$ may be the same for all $i$ (synchronous carrier - SynC) or be a random variable uniformly distributed in the interval $0 - 2\pi$ (asynchronous carrier - AsynC).

**Synchroneous versus asynchroneous mode of the random telegraph signal**

Note that the samples of the base-band random telegraph signal $b_i(t)$ can be generated in synchronous time grid ($SynT$) or in an asynchronous time grid ($AsynT$) with transitions from the states 0 and 1 at points uniformly distributed within the elementary slot of duration $T$.

**Four options of the random process with sample functions given by (25)**

Therefore, there are four options in defining the random process $\{s\}$: $SynT-SynC$, $AsynT-SynC$, $SynT-AsynC$ and $AsynT-AsynC$. The properties of the options are not the same. Therefore, any information about the properties of a PSK random process missing the information about the option is incomplete.

**Derivation of the correlation functions**

A sample of the correlation product is [12], [13]

$$r_{ss}^{(i)} = s_i(t + 0.5\tau) s_i^*(t - 0.5\tau) = e^{j\pi[\theta_i(t+0.5\tau)-\theta_i(t-0.5\tau)]} e^{j2\pi f_c \tau}. \quad (30)$$

The correlation function is

$$r_{ss}^{}(t, \tau) = E\left\{e^{j\pi[\theta(t+0.5\tau)-\theta(t-0.5\tau)]}\right\} e^{j2\pi f_c \tau}. \quad (31)$$
Computer calculations show that \( E \left\{ \cos \left[ \pi \{ b(t + 0.5\tau) \mp b(t - 0.5\tau) \} \right] \right\} \) yield a waveform depending on the choice of the case SynT or AsynT and \( E \left\{ \sin \left[ \pi \{ b(t + 0.5\tau) \mp b(t - 0.5\tau) \} \right] \right\} = 0 \).

In the case AsynT we have

\[
E \left\{ e^{j \pi \left[ b(t + 0.5\tau) \mp b(t - 0.5\tau) \right]} \right\} = \text{tri} \left( \tau / T \right),
\]

where \( \text{tri}(\sigma T) \) is a triangle function with a support from \(-T\) to \(T\). It yields

\[
T_{c1} \left( t, f \right) = \int_{-\tau}^{\tau} \text{tri} \left( \tau / T \right) e^{j \tau / T} e^{-j 2\pi f / T} d\tau = T \left[ \frac{\sin \left[ \pi \left( f - f_c \right) T \right]}{\pi \left( f - f_c \right) T} \right]^2 \otimes 1, \quad (32).
\]

The multiplication \( \otimes 1 \) indicates the independence on time. In the case SynT (32) is replaced by

\[
E \left\{ e^{j \pi \left[ b(t + 0.5\tau) \mp b(t - 0.5\tau) \right]} \right\} = \Pi_T \left( \tau / T \right),
\]

where \( \Pi_T \left( \tau / T \right) \) is a rectangular function with a support from \(-T\) to \(T\). It yields

\[
T_{c1} \left( t, f \right) = \int_{-\tau}^{\tau} e^{j \tau / T} e^{-j 2\pi f / T} d\tau = T \left[ \frac{\sin \left[ 2\pi \left( f - f_c \right) T \right]}{2\pi \left( f - f_c \right) T} \right] \otimes 1, \quad (33).
\]

**Experimental evidence that the 2PSK process, case SynT-SynC, is improper**

![Fig.13. WDs of PSK signal, option SynT-SynC, \( f_c = 1 \). a) \( T_{even} \left( t, f \right) \). b) The part \( T_{even} \). c) The part \( T_{cross} \).](image)

![Fig.14. PSK, the slices of \( T_{c1} \left( t, f \right) \). a) \( T_{c1} \left( t = 20.5, f \right) \). Solid line \( T_{even} \), dotted line, \( T_{cross} \). b) \( T_{cross} \left( t = 0 \right) \). c) A fragment of the t-axis shows details of the waveform modulating \( T_{even} \left( t, f \right) \).](image)
Fig. 15. PSK, the same slices as in Fig. 14a and c for a doubled value of the elementary time slot $T$.

Fig. 13. shows the ensemble averages of the WDs of 2PSK (2 denotes binary $b(t)$) calculated using 2000 samples. In this example the supports of the two terms of $T_{\text{even}}$ and of $T_{\text{cross}}$ do not overlap. The finite value of $T_{\text{cross}}$ (Fig. 13c) shows that the 2PSK process, case $\text{SynT-SynC}$, is improper. The slices from Fig. 14c and Fig. 15b show that $T_{\text{cross}}$ is modulated by a periodic signal. The waveform of this signal changes with the change of the elementary time slot of the transmitted random telegraph signal, $b(t)$. Concluding, in the case $\text{SynC-SynT}$ the cross-term yields the information about the carrier frequency and the length of the elementary time slot (bit rate), compare Fig. 14c and 15b.

**Computer simulations, case 2PSK, AsynT-SynC.**

The result of computer simulations is shown in Fig. 16. The comparison of Figs 14 and 15 with Fig. 16 shows that in the case 2PSK, AsynT–SynC, the waveform modulating the cross-term is a pure sine.

The information about the length of the time slot $T$ is lost. Note that the eventual change from the case SynC to AsynC cancels the cross-terms.

Fig. 16. 2PSK, case AsynT-SynC. a) $T_{xx}(t, f)$, b) The slices of $T_{xx}(t = 20, f)$, $T_{\text{even}}(t = 20, f)$ (solid line), and $T_{\text{cross}}(t = 20, f)$ (dotted line), c) The slice $T_{\text{cross}}(t, f = 0)$ shows a sine waveform instead of the distorted waveforms of Fig. 14 and 15. The information about the length of the time slot $T$ is lost.

**32PSK, case SynT-SynC.**
The properties of ensemble averages of the WDs of random processes with sample functions defined by $M$-nary PSK differ in comparison to the binary case. Fig.17 shows the correlation functions $r_{ss}(t,\tau) = r_{ss}(t,\tau)$ calculated for 32PSK, case SynT-SynC, using 2000 samples. Evidently, their difference equals nearly zero, i.e., the process is proper. Fig.18 shows the corresponding $T_{ss}(t,f) = T_{\text{even}}$ (the level of $T_{\text{cross}}$ is negligible). The slices of Fig.18b) and c) show that $T_{\text{even}}$ is unipolar and modulated by a periodic signal.

**Fig.17.** 32PSK, case SynT-SynC. Terms of the correlation function defined by (17). a) $r_{ss}(t,\tau)$, b) $r_{ss}(t,\tau)$ and c) $r_{ss} - r_{ss}$. Evidently $r_{ss} = r_{ss}$ and the 32PSK process is proper.

**Fig.18.** 32PSK. a) The ensemble average $T_{ss} = T_{\text{even}}$ (the level of $T_{\text{cross}}$ is negligible), b) The slice $T_{\text{even}}(t,f = \pm 1)$ and c) A fragment of the $t$-axis of b).

The binary frequency shift random process is cyclostationary

A sample of the FSK analytic (carrier frequency $f_c$ sufficiently large) random process is given by the formula

$$s_i(t) = e^{[2\pi(f_c + \Delta f_c,\omega_{ei}(t)) + \theta]},$$  \hspace{1cm} (36)

where $b_{ji} = -0.5 + b_j(t)$ yields a symmetrical frequency shift from $-0.5\Delta f_c$ to $0.5\Delta f_c$. Again, the four options defined for the PSK can be applied. This is a specific example. WDs of a single sample are shown in Fig.19. The supports of $W_{\text{even}}$ and $W_{\text{cross}}$ are disjoint. We observe that $W_{\text{even}}$ is a random field. However, the random field of $W_{\text{cross}}$ contains deterministic periodic terms. Fig.20 presents the
estimates of the correlation functions defined by (17). Both terms \( r_{xx}(t, \tau) \) and \( r_{\tilde{x}}(t, \tau) \) are periodic functions w.r.t. both variables, i.e., the global time \( t \) and the local time shift \( \tau \). The periodicity w.r.t. \( \tau \) shows that the 2FSK process is cyclostationary. Fig.20c yields the evidence that \( r_{xx} \neq r_{\tilde{x}} \), i.e., the process is improper. The estimate of the ensemble average of the WD calculated for 1000 samples is shown in Fig.21a. The slice of the cross term in Fig.21b shows the nature of the modulation of the cross term. A periodic wave of fundamental frequency equal \( 2f_c \) has an envelope. The frequency of this envelope equals the double value of the frequency shift \( \Delta f_c \). The slice of \( T_{\text{even}} \) in Fig.21c has the form of a Gabor wavelet. We observe here the term \( T_{\text{even}} \) is here a bipolar function of \( t \). The width of the main period of the wavelet depends on the value of the frequency shift. Again, the change from the case SynC to A synC cancels the cross-term.

Fig.19. 2FSK, case SynT-SynC, WDs of a single sample, carrier frequency \( f_c = 1 \), frequency deviation \( \Delta f_c = 0.1 \). a) \( W_{xx} \), b) \( W_{\text{even}} \) and c) \( W_{\text{cross}} \).

Fig.20. 2FSK, case SynT-SynC. Terms of the correlation function defined by (17). a) \( r_{xx}(t, \tau) \), b) \( r_{\tilde{x}}(t, \tau) \) and c) \( r_{xx} - r_{\tilde{x}} \). Evidently \( r_{xx} \neq r_{\tilde{x}} \) are periodic functions w.r.t. \( t \) and \( \tau \).
Fig. 21. 2FSK, case SynT-SynC, \( f_c = 1, \Delta f_c = 0 \). a) The estimate of the ensemble average \( T_{ss} (t, f) = 0.5T_{\text{even}} + 0.25T_{\text{cross}} \), b) The slice \( T_{\text{cross}} (t, f = 0) \) and c) The slice \( T_{\text{even}} (t, f = \pm 1) \).

The M-nary PSK and FSK have equal ensemble averages of the Wigner distributions and are proper.

However, only the 2FSK process is improper. For the M-FSK process, \( M \geq 8, r_{ss} (t, \tau) = r_{ss} (t, \tau) \) and both are functions only of \( \tau \). Therefore, the process is proper and the cross-term vanish. We found that the images in Fig. 18 and 19 calculated for 32PSK are practically the same for 32FSK.

CONCLUSIONS

1. We confirmed in several examples that WDs of a single sample of a random process have the form of a bipolar random field and that their ensemble averages are well defined deterministic functions.

2. From the point of view of illustrative presentation and computational efficiency it was convenient to calculate the WDs of real signals in the form \( W_{ss} (t, f) = 0.5W_{\text{even}} + 0.25W_{\text{cross}} \) or for ensemble averages \( T_{ss} (t, f) = 0.5T_{\text{even}} + 0.25T_{\text{cross}} \) having in mind that the parts in the half-plane \( f < 0 \) are redundant. This representation enables the presentation of important slices \( T_{\text{cross}} (t, f = 0) \). If the supports of \( W_{\text{even}} \) and \( W_{\text{cross}} \) overlap, they can be calculated separately using (3) and (4).

3. The equation (20) shows that the energy of cross-terms equals zero.

4. The double value of the real part of the complementary WD defined in [5] coincides with the cross-term \( T_{\text{cross}} \). In consequence, any statements about the role of the complementary WD apply for the statements about the role of cross-terms.

5. The 2PSK and 2FSK processes, cases SynT-SynC and AsynT-SynC, are improper. Differently, the corresponding M-PSK and M-FSK processes, \( M \geq 8 \), are proper (the cross-terms vanish). Computer calculations show, that for \( M = 32 \), the ensemble averages of WDs of PSK and FSK processes are almost the same.

6. For many processes, for example stationary and nonstationary Gaussian noise and all processes with harmonic carriers of uniformly distributed random phase, the cross-terms vanish. In computer simulations, the level of these cross-terms decreases with increasing number of samples.
7. For selected processes, the ensemble average of the cross-terms contain information about the features of the process not included in the even part. For example, for 2PSK, option SynT-SynC, the cross-term yield the information about the length of the elementary time-slot of the random telegraph signal. In the case 2FSK, option SynT-SynC, the cross-terms yield the information about the value of the frequency shift. However, the same information is included in the even part.

8. Having in mind the points 5, 6 and 7, we believe that the conclusion of the authors of [5] that the complementary WD matters in stochastic time-frequency analysis using the analytic signal should be replaced by the statement in which cases it matters.

9. Due to the large number of samples required to calculate the estimate of ensemble averages, in all cases when the theoretical model based on experimental data cannot be developed, the calculation of ensemble averages is impossible.

10. Let us have a following comment about the role of cross-terms. The instantaneous frequency of a single sample of the random process is given in terms of the WD by the formula

\[
f(t) = \frac{\int_{-\infty}^{\infty} fW_{ss}(t, f) \, df}{\int_{-\infty}^{\infty} W_{ss}(t, f) \, df}.
\]  

The instantaneous phase is given by the integral

\[
\phi(t) = \int f(t) \, dt + \phi_0,
\]  

where \(\phi_0\) is the integration constant. The information about \(\phi_0\) is not included in \(W_{ss}(t, f)\). In the case of the simple harmonic signal \(x(t) = \cos(2\pi f_c t + \phi_0)\), it may be recovered from the cross-term \(2\cos(4\pi f_c t + \phi_0)\). However, \(x(t) = \cos(2\pi f_c t + \phi_0)\) defines a random process only, if \(\phi_0\) is a random variable. And if this variable is uniformly distributed in the interval 0 -2\(\pi\), the cross term vanishes. Note that the ensemble average of the instantaneous frequency can not be calculated by inserting in (37) \(T_{ss}\) instead of \(W_{ss}\). We believe that the calculation of ensemble averages of instantaneous frequency has no sense.
Concluding remark: Investigations of ensemble averages of WDs of random processes give a deeper insight to the properties of the processes. However, their significance for practical applications could be questioned.

REFERENCES