WARSAW UNIVERSITY TECHNOLOGY
FACULTY OF ELECTRONICS
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INSTITUTE OF RADIOELECTRONICS

Report No. 5, 2002

Wigner Distributions of Complex Multidimensional Analytic Signals

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Warsaw, October 2002

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Abstract – This paper presents the notion of the 2n-D Wigner distribution (WD) of n-D complex signals using the notion of complex signals with single orthant spectra. The number of orthants of the n-D signal space $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and of the corresponding Fourier frequencies space $\boldsymbol{f} = (f_1, f_2, \dots, f_n)$ equals 2^n. However, a real signal u(x) can be reconstructed using the spectrum given in a half Fourier frequencies space. In consequence, a real signal can be reconstructed using 2^{n-1} complex signals, one for n = 1, two for n = 2, four for n = 3, etc. Since each complex signal $\psi_q(\boldsymbol{x})$ (q - the label of an orthant) defines a 2n-D WD denoted $W_q(\boldsymbol{x}, \boldsymbol{f})$, we have a single 2-D Wigner-Ville distribution (WVD) for n = 1, a set of two 4-D WD for n = 2, a set of four 6-D WD for n = 3, etc. Selected features of these distributions are discused and compared with the 2-D case. Using these distributions as weighting functions in averaging in the frequency domain the products $(2\pi f_1)^k(2\pi f_2)^l(2\pi f_3)^m$... we define moments denoted $m_{klm\cdots}(\boldsymbol{x})$ (functions of $\boldsymbol{x}$) and in averaging in the signal domain the products $x_1^k x_2^l x_3^m$ ... we define moments denoted $M_{klm\cdots}(\boldsymbol{f})$ (functions of $\boldsymbol{f}$). The significance of these moments is explained. Using the signal domain average of the distribution $W_q(\boldsymbol{x}, \boldsymbol{f})$, a complex correlation function with a single orthant power spectrum is derived. This extends the Wiener-Khinchine theorem for signals with single orthant power spectra.

1. Introduction

Nonstationary signals with time-varying spectra such as human speech, sonar and radar echos, various signals from transducers in measurements and others cannot be effectively analysed using the classical Fourier transform. This was the reason for the introduction of many time-frequency signal representations. The best known examples are the short-time Fourier transformation known as the spectrogram, the Wigner-Ville distribution (WVD) or the ambiguity function. As well, time-scale representations especially using wavelet transforms have been widely used. Several excellent research and review papers describing the theory and applications of time-frequency analysis has been printed recently, for example [1], [2], [3]. The time-frequency representations are in most cases limited to the 1-D signals, usually time signals. However, in many applications there is a need to analyse 2-D signals, for example images or 3-D signals (a sequence of images). As in the 1-D case, the n-dimensional Fourier spectral analysis translates the information about the signal from the signal space $\boldsymbol{x} = (x_1, x_2,.., x_n)$ to the Fourier frequencies space $\boldsymbol{f} = (f_1, f_2,.., f_n)$. Since the signals in the $\boldsymbol{x}$ space can be nonstationary, for the same reasons for which time-frequency distributions are applied, signal space-frequency space distributions can be applied to study the properties of n-D signals. Let us introduce the abbreviation $\boldsymbol{x}$ - $\boldsymbol{f}$ distribution. However, this paper is restricted to the $\boldsymbol{x}$ - $\boldsymbol{f}$ distributions being the n-D extension of the Wigner-Ville distribution denoted by the acronym WD. Wigner has defined a distribution for complex signals in 1932 [4]. In 1948 Ville extended this definition for 1-D complex analytic signals [5]. We found the extension of the definition of the Wigner distribution for 2-D real signals in [6]. The presented in this paper extension of the WVD is based on the definition of the multidimensional complex signals with single-orthtant spectra presented in [7], [9]. These signals are the generalization of the well-known 1-D Gabor's analytic signal.

2. The definition of n-D complex signals

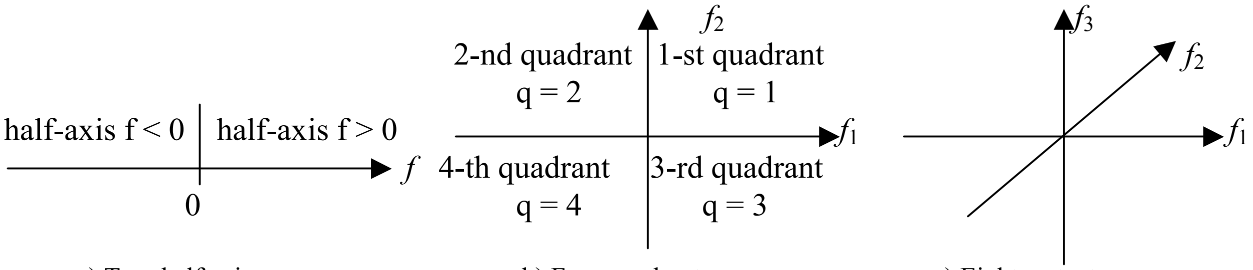
In this paper we define the notion of the WD of n-D complex signals. The starting point is the definition of the n-D Fourier transform of the n-D real signal u($\boldsymbol{x}$), where $\boldsymbol{x}$=(x_1, x_2,..., x_n) of the form

$$U(\boldsymbol{f})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}u(\boldsymbol{x})e^{-j2\pi(x_1f_1+x_2f_2+...+x_nf_n)}dx_1dx_2...dx_n \qquad (1)$$

where $\boldsymbol{f}$ = (f_1, f_2,..., f_n) defines the n-D Cartesian Fourier frequency space. Let us point out that due to the Hermitian symmetry of the spectrum U($\boldsymbol{f}$) the signal u($\boldsymbol{x}$) can be reconstructed from the knowledge of the spectrum U($\boldsymbol{f}$) in a half Fourier frequencies space. For example, the 1-D analytic signal $\psi(t)$ = u(t) + jv(t), where u and v are time signals being a pair of Hilbert transforms, is defined by the one-sided spectrum and the real signal can be reconstructed using the generalized Euler's formula

$$u(t)=0.5[\psi(t)+\psi^{*}(t)] \qquad (2)$$

where * denotes the complex conjugate. For 2-D signals u(x_1, x_2) the spectrum U(f_1, f_2) is defined in four quadrants (see Fig.1).



a) Two-half axis. b) Four quadrants c) Eight octants

Fig.1. The Fourier spaces: a) 1-D, b) 2-D and c) 3-D. In 3-D the following labeling of octants is applied: q=1 → (f_1, f_2,f_3), : q=2→ (-f_1,f_2,f_3), : q=3 → (f_1,-f_2,f_3), : q=4 → (-f_1,-f_2,f_3), : q=5 → (f_1,f_2,-f_3), : q=6 → (-f_1,f_2,-f_3), : q=7 → (f_1, -f_2,-f_3), : q=8 → (-f_1, -f_2,-f_3),

The inverse Fourier transform of the succesive single-quadrant spectra defines four complex signals ψ_1(x_1, x_2), ψ_2(x_1, x_2), ψ_3(x_1,x_2) and ψ_4(x_1, x_2). However, due to the Hermitian symmetry $\psi_1 = \psi_4^*$ and $\psi_3 = \psi_2^*$ and the reconstruction formula has the form [7], [8], [9]

$$u(x_1, x_2) = 0.25\left[\psi_1 + \psi_1^* + \psi_3 + \psi_3^*\right] \qquad (3)$$

which shows that u(x_1, x_2) is uniquely defined by a two-quadrant spectrum (half Fourier frequencies space), for example the half-space f_1 > 0. Let us denote the signals in (3) by ψ_q, where q = 1 or 3 denotes the label of the quadrant No.1 or No.3. Similarly, the spectrum of a 3-D real signal u(x_1, x_2, x_3) is defined in eight octants and the inverse Fourier transform of the single-octants spectra defines four pairs of complex signals $\psi_1(x_1,x_2,x_3)=\psi_8^*(x_1,x_2,x_3)$, $\psi_2(x_1,x_2,x_3)=\psi_7^*(x_1,x_2,x_3)$, $\psi_3(x_1,x_2,x_3)=\psi_6^*(x_1,x_2,x_3)$, $\psi_4(x_1,x_2,x_3)=\psi_5^*(x_1,x_2,x_3)$ and the reconstruction formula has the form [7], [9]

$$\mathrm{u}(x_1, x_2, x_3) = 0.125\left(\psi_1+\psi_1^*+\psi_2+\psi_2^*+\psi_3+\psi_3^*+\psi_4+\psi_4^*\right) \tag{4}$$

showing, that the signal can be reconstructed using the knowledge of the spectrum in four octants (half space) (the labels q = 1, 2, 3 or 4). This is a general property valid for higher dimensions. However, if the function u($\boldsymbol{x}$) is separable (is a product of 1-D functions) the signal is defined by the knowledge of a single orthant spectrum. Fig.7a (see point 3.1.3) shows the contour lines of a 2-D separable Gaussian signal showing the full symmetry in all four quadrants. In consequence all four complex signals have the same energy and contain the same information about the real signal. However, the Fourier spectrum is relative and may change with the shift or rotation of the Cartesian coordinate system. Especially, the rotation of the coordinate system changes a separable signal into a nonseparable one (except signals with rotational symmetry). Therefore, the rotation may cancel the symmetry. An example is shown in Fig.7b displaying the contour lines of a nonseparable Gaussian signal. This rotation produced a nonseparable signal. Here the information about the real signal is the same in the quadrants 1 and 4 and differ from the information in the quadrants 2 and 3. In consequence we have two pairs of conjugate complex signals having different energy.

3. The definition of the WD of n-D complex signals

The definition of the Wigner Distribution (WD) of n-D complex signals is a generalization of the definition of the WVD of 1-D analytic signals. The WVD of $\psi(t)$ has the form of the Fourier transform of the correlation product

$$r(t,\tau)=\psi(t+\tau/2)\psi^*(t-\tau/2) \tag{5}$$

Alternatively, the WVD may be defined by the inverse Fourier transformation of the frequency domain correlation product

$$\mathrm{g}(f,\mu)=\Gamma(f+\mu/2)\Gamma^*(f-\mu/2) \tag{6}$$

where $\Gamma(f)$ is the one-sided spectrum of $\psi(t)$. Concluding, the Wigner-Ville time-frequency distribution is defined by the Fourier transforms

$$\mathrm{W}(t,f)=\mathrm{F}[\mathrm{r}(t,\tau)]=\int_{-\infty}^{\infty}\psi(t+\tau/2)\psi^*(t-\tau/2)e^{-j2\pi f\tau}\mathrm{d}\tau \tag{7}$$

$$\mathrm{W}(t,f)=\mathrm{F}^{-1}[\mathrm{g}(f,\mu)]=\int_{-\infty}^{\infty}\Gamma(f+\mu/2)\Gamma^*(f-\mu/2)e^{j2\pi\mu t}\mathrm{d}\mu \tag{8}$$

The equations (7) and (8) define exactly the same 2-D function and the choice is a matter of convenience in given circumstances. The WD of an n-D complex signal with a single orthant spectrum is defined here by the Fourier transform of the correlation product

$$r_q\left(x,f\right)=\psi_q(\alpha_+)\psi_q^*(\alpha_-) \tag{9}$$

where $\alpha_+=\left(x_1+\chi_1/2,x_2+\chi_2/2,\cdots,x_n+\chi_n/2\right)$, $\alpha_-=\left(x_1-\chi_1/2,x_2-\chi_2/2,\cdots,x_n-\chi_n/2\right)$ and χ = (χ_1, χ_2, ...,χ_n) is the $\boldsymbol{x}$ domain shift variable. Therefore, the WD is given by the Fourier transform

$$\mathrm{W_q}(\boldsymbol{x},\boldsymbol{f})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\psi_q(\alpha_+)\psi_q^*(\alpha_-)e^{-\mathrm{j}2\pi(\chi_1 f_1+\chi_2 f_2+\cdots+\chi_n f_n)}\mathrm{d}\chi_1\mathrm{d}\chi_2\cdots\mathrm{d}\chi_n \tag{10}$$

This is a generalization of Eq.(7). The generalization of Eq.(8) is given by the inverse Fourier transform of the correlation product

$$g_\mathrm{q}(\boldsymbol{x},\ \boldsymbol{f})=\Gamma_\mathrm{q}(\beta_+)\Gamma_\mathrm{q}^*(\beta_-) \tag{11}$$

where $\Gamma_\mathrm{q}(\boldsymbol{f})$ is the single orthant spectrum of the complex signal $\psi_\mathrm{q}(\boldsymbol{x})$ We used the notation $\beta_+=\left(f_1+\mu_1/2,f_2+\mu_2/2,\cdots,f_n+\mu_n/2\right)$, $\beta_-=\left(f_1-\mu_1/2,f_2-\mu_2/2,\cdots,f_n-\mu_n/2\right)$, where $\boldsymbol{\mu}$ = (μ_1, μ_2, …, μ_n) is the frequency domain shift variable. Therefore,

$$\mathrm{W_q}(\boldsymbol{x},\ \boldsymbol{f})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Gamma_q(\beta_+)\Gamma_q^*(\beta_-)e^{\mathrm{j}2\pi(x_1\mu_1+x_2\mu_2+\cdots+x_n\mu_n)}\mathrm{d}\mu_1\mathrm{d}\mu_2\cdots\mathrm{d}\mu_n \tag{12}$$

Again, (10) and (12) define exactly the same distribution and the choice depends on circumstances. Since the half Fourier space has 2^{n-1} orthants, and each single orthant spectrum defines a different complex signal (except the case of separable signals) and we have to define 2^{n-1} different distributions. Therefore 2-D signals are described by two distributions denoted $W_1(x_1, x_2, f_1, f_2)$ and $W_3(x_1, x_2, f_1, f_2)$ and 3-D signals by four distributions denoted $W_1(x_1, x_2, x_3, f_1, f_2, f_3)$, $W_2(x_1, x_2, x_3, f_1, f_2, f_3)$, $W_3(x_1, x_2, x_3, f_1, f_2, f_3)$, and $W_4(x_1, x_2, x_3, f_1, f_2, f_3)$, each defined by the integral (10) or (12) by the addition of an appropriate subscript to ψ or Γ.

3.1 *Properties of the* **WVD** *of 1-D and the* **WD** *of 2-D signals.*

The properties of the WVD of 1-D analytic signals are well known and described in many papers and handbooks. Extensive bibliography can be found in [1] and [3]. Descriptions for 1-D signals presented below serve as a starting point for presenting the features of the WD of 2-D complex signals.

3.1.1 *The distribution* **W(*t, f*)** *of analytic signals is one-sided in the half-plane* ***f* > 0.**

Let us emphasise that the WVD of analytic signals is a one-sided function of the frequency f, that is it vanishes for $f < 0$. This is an important property caused by the one-sided spectrum of the analytic signal. Especially, due to this property the WVD of analytic signals do not generate the cross terms around the zero frequencies (see next part of this paper). Let us illustrate this property with examples.

1. The WVD of the harmonic analytic signal $\psi(t) = e^{j2\pi f_0 t}$ has the form W(t, f) = $\delta(f - f_0)$. Notice that this is a function of both variables and formally could be written in the form of a product of distributions W(t, f) = $\delta(f - f_0)\otimes\mathbf{1}$, where **1** equals one for all t (see Fig.2). However, this notation which shows the time dependence will not be used in this paper.

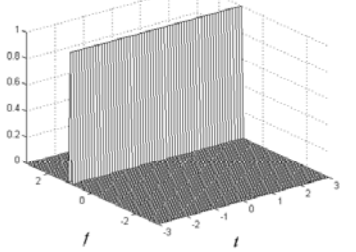


Fig.2. Simulation of the WVD W(t, f) = $\delta(f - f_0)$, $f_0 = 1$.

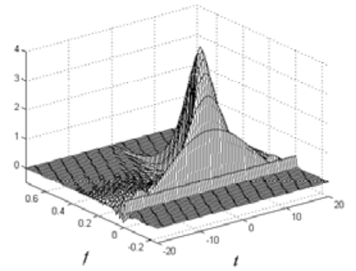


Fig.3. The WVD of the Cauchy analytic signal.

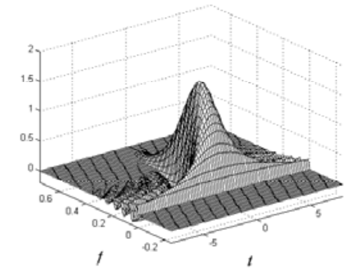


Fig.4. The WVD of the Gaussian analytic signal.

2. The one-sided WVD of the Cauchy analytic signal $\psi(t) = (1/\pi)(a - jt)^{-1}$.is shown in Fig.3.

3. Fig.4 shows the one-sided WVD of the Gaussian analytic signal $\psi(t) = e^{-\pi t^2} + j\mathrm{H}\left[e^{-\pi t^2}\right]$ where H denotes the Hilbert transform. Formally, it is possible to define the distribution W(t, f) using the conjugate kernel $r^*(t,\tau) = \psi^*(t+\tau/2)\psi(t-\tau/2)$. Such a distribution is one-sided in the half-plane $f < 0$. If needed, we shall denote the distribution (7) defined by the kernel (5) by $W_+(t, f)$ and by the conjugate kernel by $W_-(t, f)$. This distiction is useful in the description of WD of 2-D separable complex signals.

3.1.2 ***Single-quadrant nature of the cross-section of the* WD *of 2-D signals***

The WD of a 2-D complex signal is a 4-D function. Since the real signal u(x_1, x_2) defines a set of two complex signals with single-quadrant spectra denoted $\psi_1(x_1, x_2)$ and $\psi_3(x_1, x_2)$ the Eq.(10) defines a set of two four-dimensional Wigner distributions denoted $W_1(x_1, x_2, f_1, f_2)$ and $W_3(x_1, x_2, f_1, f_2)$. Notice, that the subscripts 1 and 3 correspond to the quadrants No.1 and No.3 of the $\boldsymbol{f}$ plane. Let us define cross-sections of W_1 and W_3 for fixed values of the variables $x_1 = x_{10}$ and $x_2 = x_{20}$. The one-sided distributions of Fig.2 to 4 correspond here to single-quadrant cross-sections $W_1(x_{10}, x_{20}, f_1, f_2)$ and $W_3(x_{10}, x_{20}, f_1, f_2)$. Let us illustrate this statement with the following examples:

1. Consider the real signal u(x_1, x_2) = $\cos(2\pi f_a x_1)\cos(2\pi f_b x_2)$. The corresponding complex signal defined by the spectrum located in the first quadrant has the form $\psi_1(x_1, x_2) = e^{j2\pi(x_1 f_a + x_2 f_b)}$. Since the signal is separable, there is no need to use the signal ψ_3. The WD of this signal has the form $W_1(x_1, x_2, f_1, f_2) = \delta(f_1 - f_a)\otimes(f_2 - f_b)$ for all $\boldsymbol{x}$. Fig.5 shows the cross-section of this distribution for $x_1 = x_2 = 0$. Notice, that this cross-section does not depend of the values on the fixed variables x_{10} and x_{20}.

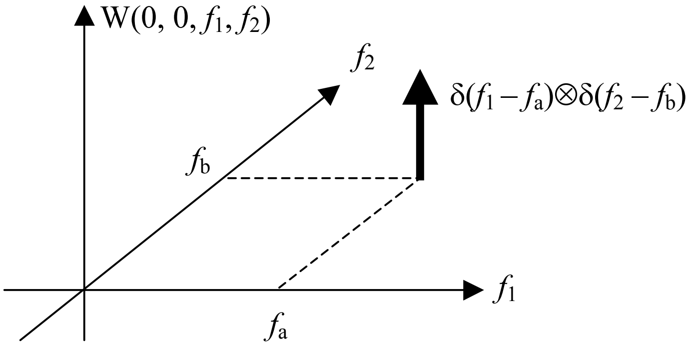
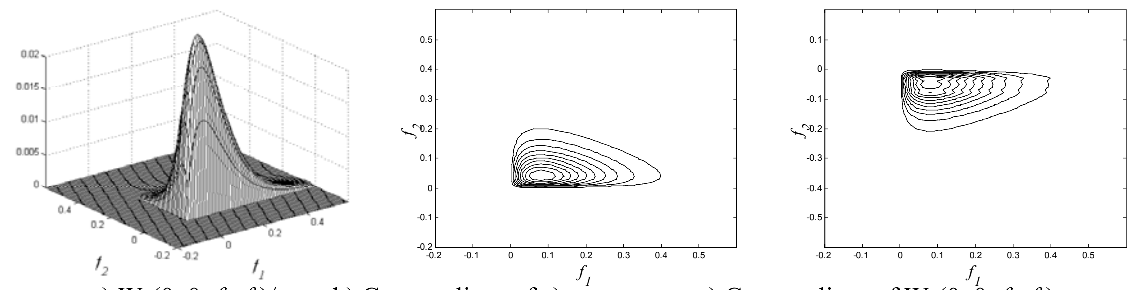


Fig.5. The cross-section of the 4-D WD of the signal u((x_1, x_2) = cos(2π$f_a x_1$) cos(2π$f_b x_2$).

2. Consider the separable 2-D Cauchy signal of the form

$$\mathrm{u}(x_1, x_2) = \frac{1}{\pi^2} \frac{\mathrm{a}}{\mathrm{a}^2 + x_1^2} \frac{\mathrm{b}}{\mathrm{b}^2 + x_2^2} \tag{13}$$

Fig.6a shows the single-quadrant cross-section of the distribution $W_1(x_1, x_2, f_1, f_2)$ and Fig.6b the corresponding contour lines. The contour lines of the distribution $W_3(x_1, x_2, f_1, f_2)$ are shown in Fig.6c. The examination of the contour lines shows that the information contained in both cross-sections is the same as expected for a separable signal. In Fig.6c we see an mirror image in respect to the line $f_1 = 0$ of the image in Fig.6b.



a) $W_1(0, 0, f_1, f_2)$/ b) Contour lines of a). c) Contour lines of $W_2(0, 0, f_1, f_2)$.

Fig.6. The cross-sections of the 4-D WD of a separable Cauchy 2-D signal. W_1 has a single-quadrant support in the first quadrant and W_2 in the third quadrant/

3. Consider the separable 2-D Gaussian signal given by the equation

$$\mathrm{u}(x_1, x_2) = \frac{1}{2\pi} \exp\left[-0.5\left(x_1^2 \mathrm{k}^2 + \frac{x_2^2}{\mathrm{k}^2} \right) \right]$$

Let us introduce the rotation of the coordinates with $x_1 = x_1' \cos(\gamma) - x_2' \sin(\gamma)$, $x_2 = x_1' \sin(\gamma) + x_2' \cos(\gamma)$.

We get

$$\mathrm{u}\left(x_1', x_2'\right) = \frac{1}{2\pi} \exp\left[-\left(\mathrm{a} x_1'^2 + \mathrm{b} x_2'^2 + \mathrm{c} x_1' x_2' \right) \right] \tag{14}$$

where $\mathrm{a}^2 = 0.5\left[\mathrm{k}^2 \cos^2(\gamma) + \frac{\sin^2(\gamma)}{\mathrm{k}^2} \right]$, $b^2 = 0.5\left[\mathrm{k}^2 \sin^2(\gamma) + \frac{\cos^2(\gamma)}{\mathrm{k}^2} \right]$ and $\mathrm{c} = \sin(\gamma)\cos(\gamma)\left(\mathrm{k}^2 - \frac{1}{\mathrm{k}^2} \right)$.

The energy of (14) does not depend on the angle of rotation γ. The spectrum is signal is

$$\mathrm{U}\left(f_1, f_2\right) = \exp\left[-\left(\mathrm{b}^2 \omega_1^2 + \mathrm{a}^2 \omega_2^2 + \mathrm{c}\,\omega_1 \omega_2 \right) \right] \quad ; \omega_1 = 2\pi f_1,\ \omega_2 = 2\pi f_2 \tag{15}$$

Fig.7a shows the contour lines for the separable signal with k = 1.5 and $\gamma = 0$ and Fig.7b for a nonseparable signal with $\gamma = \pi/3$. In the separable case the spectral information in all four quadrants is the same. In the nonseparable case the spectral information in the quadrants No.1 and 4 is different in repect to the quadrants No.2 and 3.

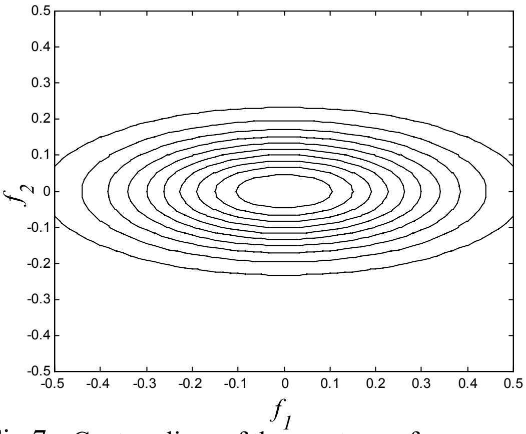


Fig.7a. Contour lines of the spectrum of a separable 2-D Gaussian signal ($\gamma = 0$). The spectral information in all quadrants is the same.

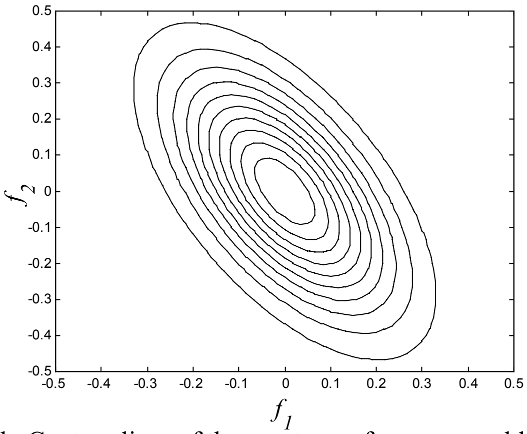


Fig.7b. Contour lines of the spectrum of a nonseparable Gaussian signal ($\gamma = \pi/3$). The spectral information is different in the quadrants No.1 and 3 and is equal in the quadrants No.1 and 4 and no.2 and 3.

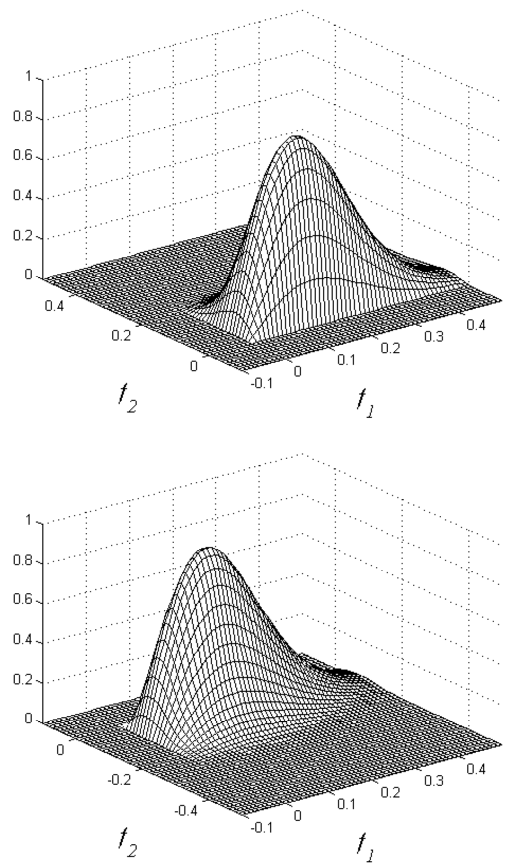


Fig.8a. The cross-sections of theWD's correspon-ding to the spectrum of Fig.7a.Upper image $W_1(0, 0, f_1, f_2)$. Bottom: $W_3(0, 0, f_1, f_2)$. W_2 is a mirror image of W_1 in respect to the line $f_2 = 0$.

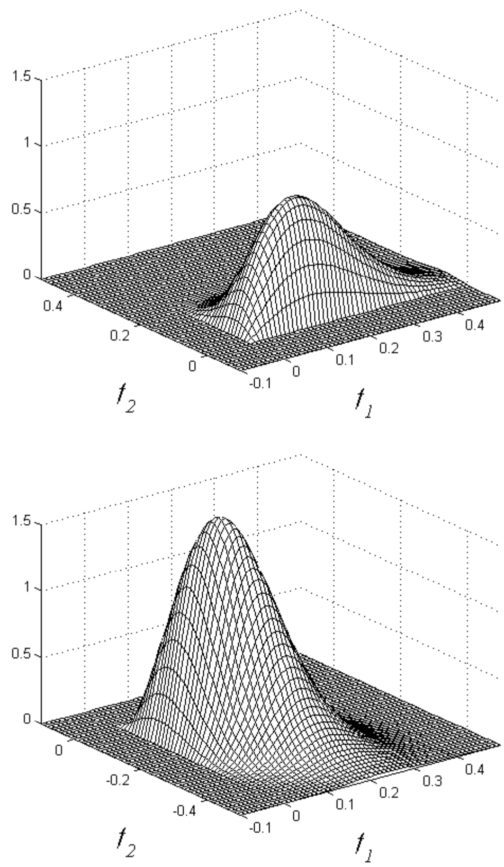


Fig.8b. The cross-sections of the WD's correspon-ding to the spectrum of Fig.7b. Upper image $W_1(0, 0, f_1, f_2)$. Bottom: $W_3(0, 0, f_1, f_2)$. W_1 and W_2 are different.

Fig.8a shows the cross-sections $W_1(0, 0, f_1, f_2)$ and $W_3(0, 0, f_1, f_2)$ for the separable case with $\gamma = 0$ (see Fig.7a) Again the W_3 is amirror image of W_1 in respect to the line $f_1 = 0$. The energy of both distribution is the same. Fig.8b shows the above cross-sections for the non-separable case with $\gamma = \pi/3$. The two cross-sections of Fig.8b as expected are different and have different energies.

3.1.3 ***Interference (cross) terms of the* WV *distributions of 1-D signals***

The main drawback of the WVD is, that the algorithm is bilinear in nature and produces cross terms. Consider a sum of two analytic signals $\psi(t)= \psi_a(t) + \psi_b(t)$. Let us denote $\psi_{a+} = \psi_a\left(t+\tau/2\right)$ and $\psi_{a-} = \psi_a\left(t-\tau/2\right)$ amd respectively ψ_{b+} and ψ_{b-}. The WVD of this signal is

$$W(t, f)=F\left[\left(\psi_{a+}+\psi_{b+}\right)\left(\psi_{a-}^{*}+\psi_{b-}^{*}\right)\right]=$$
$$F\left[\psi_{a+}\psi_{a-}^{*}\right]+F\left[\psi_{b+}\psi_{b-}^{*}\right]+F\left[\psi_{b+}\psi_{a-}^{*}+\psi_{a+}\psi_{b-}^{*}\right] \tag{17}$$

The first two terms are equal to the WVD of the signals ψ_a and ψ_b respectively and the last two represent the cross terms. Example: Consider the sum of two analytic harmonic signals $\psi\left(t\right)=e^{j2\pi f_a t}+e^{j2\pi f_b t}$. The WVD of this signal is (see Fig.9 (left))

$$W(t, f)=\delta(f - f_a)+\delta(f - f_b)+2\cos[2\pi(f_b - f_a)t]\delta[f - 0.5(f_a + f_b)] \tag{18}$$

Notice, that the oscillating cross term is represented by the delta surface positioned at the mean frequency $(f_a + f_b)/2$ and multiplied by 2 while the other deltas are multiplied by 1.

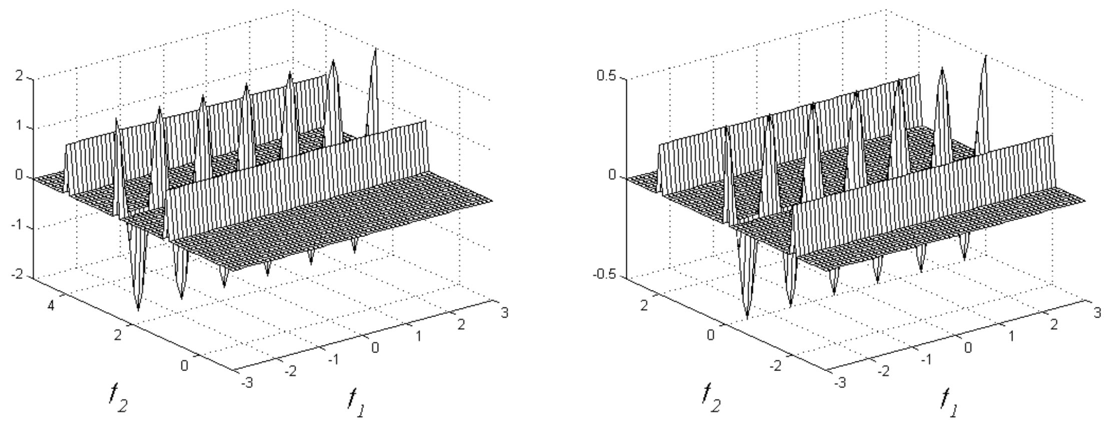


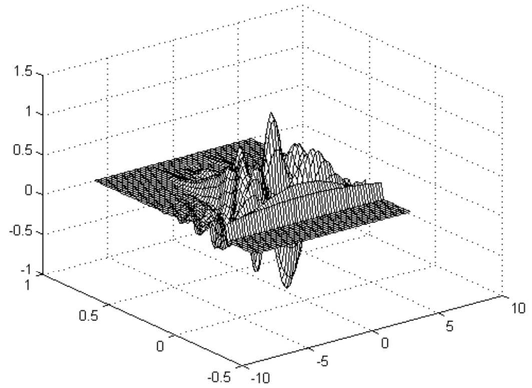
Fig.9.Left: The one-sided WVD of the analytic signal $\psi\left(t\right)=e^{j2\pi f_a t}+e^{j2\pi f_b t}$, $f_a = 1$, $f_b = 4$. The cross-terms are oscillating along the line $f = 0.5(f_a + f_b) = 2.5$.

Right: The two-sided WVD of a single real signal $u\left(t\right)=\cos\left(2\pi f_0 t\right)=\dfrac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}$, $f_0 = 2$. The cross-terms are oscillating along the zero frequency line.

Next example shows the cross terms of the WVD of a sum of two Gaussian analytic signals given by the formula

$$\psi(t)=e^{-\pi(t-t_0)^2}+e^{-\pi(t+t_0)^2}+j\,H\left[e^{-\pi(t-t_0)^2}+e^{-\pi(t+t_0)^2}\right] \tag{15}$$

The one-sided WVD of this signal is displayed in Fig.10. The comparison with Fig.8 shows the existence of the cross-terms.



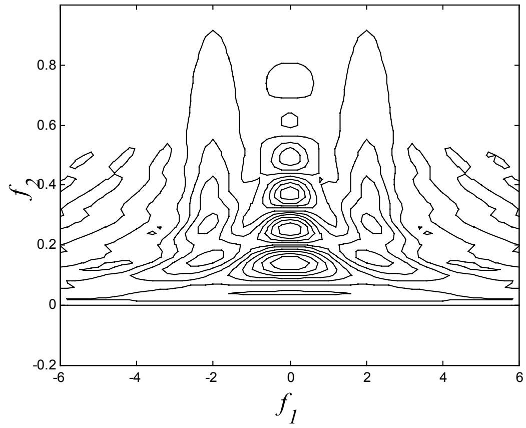


Fig.10. Left: The WVD of the signal given by the Eq.(15). Right: The contours of this WVD.

3.1.4. ***Cross terms for real signals***

For analytic signals the cross terms are generated, if the signal is a sum of at least two signals. However, a real signal can be written as a sum of two conjugate analytic signals (see Eq.(2)). Due to the interaction of the terms defined by the positive frequencies and negative frequencies parts of the spectrum cross terms are generated around the zero frequency. As an example consider the harmonic signal $\cos(2\pi f_a t)$. The WVD of this signal can be obtained by inserting in (8) $f_b = -f_a$ and dividing by 4.This yields

$$W(t, f)= 0.25[\delta(f-f_a)+\delta(f+f_a)]+0.5\cos(4\pi f_a)\delta(f) \tag{16}$$

This distribution is shown in Fig.9 (right) (compare with Fig.4).

3.1.5 ***Cross terms of the* WD *of 2-D signals***

As in the case of the WVD of 1-D signals the WD of 2-D signals generates cross terms. The derivation is the same as given by the Eq.(7). Consider the sum of two complex signals

$$\psi(x_1, x_2) = \psi_a(x_1, x_2) + \psi_b(x_1, x_2) \tag{17}$$

The WD of this signal has the form

$$W_1(x_1, x_2, f_1, f_2) = F\left[\psi_{a+}\psi^*_{a-}\right] + F\left[\psi_{b+}\psi^*_{b-}\right] + F\left[\psi_{a+}\psi^*_{b-} + \psi_{b+}\psi^*_{a-}\right] \tag{18}$$

The last term represents the cross terms. Example: Consider the complex signal

$$\psi(x_1, x_2) = e^{j2\pi(f_{1a}x_1 + f_{2a}x_2)} + e^{j2\pi(f_{1b}x_1 + f_{2b}x_2)} \tag{19}$$

The insertion of this signal in (18) and the evaluation of the Fourier transforms yields

$$\begin{aligned} W_1(x_1, x_2, f_1, f_2) &= \delta(f_1 - f_{1a}) \otimes \delta(f_2 - f_{2a}) + \delta(f_1 - f_{1b}) \otimes \delta(f_2 - f_{2b}) \\ &+ 2\cos\left\{2\pi\left[\left(f_{1a} - f_{1b}\right)x_1 + \left(f_{2a} - f_{2b}\right)x_2\right]\right\}\delta\left(f_1 - \frac{f_{1a} + f_{1b}}{2}\right) \otimes \delta\left(f_2 - \frac{f_{2a} + f_{2b}}{2}\right) \end{aligned} \tag{20}$$

Notice, that the deltas should be formally multipled by the functions of (x_1, x_2) equal one for all x_1, x_2.

3.1.6 ***Cross terms of* WD *of 2-D real signals***

The real 2-D signal is represented in terms of the complex signals by the Eq.(3). Therefore, the WD is given by the Fourier transform

$$\mathrm{W}_1(x_1, x_2, f_1, f_2) = \frac{1}{16}\mathrm{F}\left\{\left[\psi_{1+} + \psi_{1+}^* + \psi_{3+} + \psi_{3+}^*\right]\left[\psi_{1-} + \psi_{1-}^* + \psi_{3-} + \psi_{3-}^*\right]\right\} \tag{21}$$

The multiplication of the terms in paranthesis yields 16 terms including the cross terms. Let us have an example with the signal $\mathrm{u}(x_1, x_2) = \cos(2\pi f_a x_1)\cos(2\pi f_b x_2)$. The complex signals ψ_1 and ψ_3 have the form

$$\psi_1(x_1, x_2) = \mathrm{e}^{j2\pi(f_a x_1 + f_b x_2)} \quad ; \quad \psi_3(x_1, x_2) = \mathrm{e}^{j2\pi(f_a x_1 - f_b x_2)} \tag{22}$$

The insertion of ψ_1 in (21) yields sixteen Fourier transforms. The appropriate summation of these transforms yields

$$\begin{aligned} \mathrm{W} = {} & (1/16)[\delta(f_1 + f_a)\otimes\delta(f_2 + f_b) + \delta(f_1 - f_a)\otimes\delta(f_2 + f_b) + \delta(f_1 + f_a)\otimes\delta(f_2 - f_b) + \delta(f_1 - f_a)\otimes\delta(f_2 - f_b)] \\ & + (1/4)\cos(4\pi f_a x_1)\cos(4\pi f_b x_2)\delta(x_1)\otimes\delta(x_2) \\ & + (1/8)\{\cos(4\pi f_a x_1)\,[\delta(f_2 + f_b) + \delta(f_2 - f_b)] + \cos(4\pi f_b x_2)\,[\delta(f_1 + f_a) + \delta(f_1 - f_a)]\} \end{aligned} \tag{23}$$

Notice, that all terms are four-dimensional functions. If the function of a variable x_1 or x_2 is not displayed it is equal to one for all x_1 or x_2 . The first four terms represent the power spectrum in all four quadrants and the next two terms represent the cross terms. Since the real signal $\mathrm{u}(x_1, x_2)$ is separable, the complex signal $\psi_1(x_1, x_2)$ contains all information about u. The WD of this complex signal has the form

$$\mathrm{W}(x_1, x_2, f_1, f_2) = \delta(f_1 - f_a)\otimes\delta(f_2 - f_b) \tag{24}$$

The comparison with (23) shows clearly the advantages of using the complex notation. However, for nonseparable signals two distributions $\mathrm{W}_1(x_1, x_2, f_1, f_2)$ and $\mathrm{W}_3(x_1, x_2, f_1, f_2)$ should be calculated.

4. Energy relations and moments

Let us write the analytic signal in the exponential form $\psi(t) = \mathrm{u}(t) + j\mathrm{v}(t) = \mathrm{A}(t)\exp[j\tilde{\Phi}(t)]$. The energy of this signal is given by the integral

$$\mathrm{E}_\psi = \int_{-\infty}^{\infty}\psi(t)\psi^*(t)\mathrm{d}t = \int_{-\infty}^{\infty}\left[u^2(t) + v^2(t)\right]dt = \int_{-\infty}^{\infty}\mathrm{A}^2(t)\mathrm{d}t = 2\mathrm{E}_\mathrm{u} \tag{25}$$

where $\mathrm{E_u} = \int \mathrm{u}^2(t)\mathrm{d}t = \int \mathrm{v}^2(t)\mathrm{d}t$, that is both terms of the analytic signal have an equal energy. The WVD may be used to define instantaneous (time dependent) spectral moments and frequency dependent time moments. The instantaneous (functions of time) spectral moments are defined by the integral

$$\mathrm{m_k}(t) = \int_{-\infty}^{\infty}(2\pi f)^{\mathrm{k}}\,\mathrm{W}(t, f)\mathrm{d}f \tag{26}$$

The k-th power of $(2\pi f)$ is averaged using the distribution $\mathrm{W}(t, f)$. The integration yields (see Appendix)

$$\mathrm{m}_k(t)=(-\mathrm{j})^k \lim_{\tau\to 0}\left\{\frac{d^{\mathrm{k}}}{d\tau^{\mathrm{k}}}\left[\psi(t+\tau/2)\psi^*(t-\tau/2)\right]\right\} \tag{27}$$

The zero order moment equals the square of the instantaneous amplitude:

$$\mathrm{m}_0(\mathrm{t})=\psi(t)\psi^*(t)=\mathrm{A}^2(t)=\mathrm{u}^2(t)+\mathrm{v}^2(t) \tag{28}$$

and is called alternatively the time marginal of the function W(t, f). The first order moment is

$$\mathrm{m}_1(t)=(\mathrm{j}/2)\left[\psi(\mathrm{t})\dot{\psi}^*(t)-\dot{\psi}(\mathrm{t})\psi^*(t)\right]=u(t)\dot{v}(t)-v(t)\dot{u}(\mathrm{t}) \tag{29}$$

The second order moment is

$$\mathrm{m}_2(t)=0.25\left[2\dot{\psi}(t)\dot{\psi}^*(\mathrm{t})-\psi(t)\psi^*(t)-\psi^*(t)\psi(t)\right]$$
$$=0.5\left[\dot{u}^2(t)+\dot{v}^2(t)-u(t)\ddot{u}(t)-v(t)\ddot{v}(t)\right] \tag{30}$$

Let us recall, that the analytic signal defines the instantaneous complex frequency [9] [8] in the form $s(t)$ = (d/dt)Log[ψ(t)]=$\alpha(t)+\mathrm{j}\omega(t)$. It can be shown, that $\alpha(t)=0.5\,\dot{\mathrm{m}}_0(t)/\mathrm{m}_0(t)$ and $\omega(t)=2\pi f(t)=\mathrm{m}_1(t)/\mathrm{m}_0(t)$.

The frequency dependent time moments are defined by the integral

$$\mathrm{M_k}(f)=\int_{-\infty}^{\infty} t^k\mathrm{W}(t,f)\mathrm{d}t \tag{31}$$

The k-th power of the variable t is averaged using the distribution W(t, f). The integration yields (see the Appendix)

$$\mathrm{M_k}(f)=\lim_{\mu\to 0}\left\{\left(\frac{\mathrm{j}}{2\pi}\right)^{\mathrm{k}}\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{d}\mu^{\mathrm{k}}}\left[\Gamma(f+\mu/2)\Gamma^*(f-\mu/2)\right]\right\} \tag{32}$$

The zero order moment yields a one-sided power spectrum of the analytic signal

$$\mathrm{M}_0(\mathrm{f})=\Gamma(f)\Gamma^*(f)=[1+\mathrm{sgn}(f)]^2\,|\mathrm{U}(f)|^2 \tag{33}$$

where $\Gamma(f)$ = [1+sgn(f)]U(f) is the one-sided Fourier spectrum of the analytic signal. The zero order moment $\mathrm{M}_0(f)$ is called alternatively the frequency marginal of the function W(t, f). The first order moment $\mathrm{M}_1(f)$ is given by the formula

$$\mathrm{M}_1(f)=\frac{\mathrm{j}}{4\pi}\left\{\frac{\mathrm{d}\Gamma(f)}{\mathrm{d}f}\Gamma^*(f)-\frac{\mathrm{d}\Gamma^*(f)}{\mathrm{d}f}\Gamma(f)\right\} \tag{34}$$

The insertion of $\Gamma(f)$ and $\mathrm{U}(f)=\mathrm{U}_{\mathrm{Re}}(f)+\mathrm{jU}_{\mathrm{Im}}(f)$) yields

$$\mathrm{M}_1(f)=\frac{1}{2\pi}[1+\mathrm{sgn}(f)]^2\left\{\frac{\mathrm{dU_{Re}}(f)}{\mathrm{d}f}\mathrm{U_{Im}}(f)-\frac{\mathrm{dU_{Im}}(f)}{\mathrm{d}f}\mathrm{U_{Re}}(f)\right\} \tag{35}$$

Notice, that for even signals $\mathrm{U_{Im}}(f)$) = 0 and for odd signals $\mathrm{U_{Re}}(f)$ = 0. In both cases $\mathrm{M}_1(f)$ = 0. In consequence, the value of $\mathrm{M}_1(f)$ is a measure of noneveness and nonoddness of a given signal.

4.1 *The complex correlation function*

Let us derive (33) again using the Wiener-Khinchine theorem. We start with the calculation of the frequency marginal of W(t, f) [3] (see Eq.33)

$$\overline{\mathrm{W}(t,f)} = \int_{-\infty}^{\infty} \mathrm{W}(t,f)\,\mathrm{d}t \qquad 36)$$

The insertion of W(t, f) given by (7) and introducing a change of the order of integration yields

$$\overline{\mathrm{W}(t.f)} = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \psi(t+\tau/2)\,\psi^{*}(t-\tau/2)\,\mathrm{d}t \right\} \mathrm{e}^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \qquad (37)$$

The limit in the paranthesis defines a complex correlation function. The insertion ψ(t) = u(t) + jv(t) yields

$$\overline{\mathrm{W}(t,f)} = \rho_{\mathrm{uu}}(\tau) + \rho_{\mathrm{vv}}(\tau) + \mathrm{j}\left[\kappa_{\mathrm{vu}}(\tau) - \kappa_{\mathrm{uv}}(\tau)\right] \qquad (38)$$

where ρ_{uu} and ρ_{vv} are autocorrelation functions and κ_{vu} and κ_{uv} crosscorrelation functions. However, it can be shown [9] that $\rho_{uu} = \rho_{vv}$ and and $\kappa_{vu} = -\kappa_{uv}$. In consequence, the limit in paranthesis in (37) defines a complex correlation function

$$\rho_c(\tau) = 2\rho_{uu}(\tau) + \mathrm{j}2\kappa_{vu}(\tau) \qquad (39)$$

It may be shown, that $\rho_c(\tau)$ is analytic, that is $\kappa_{vu} = H[\rho_{uu}]$. Therefore, since (33) is the Fourier transform of ρ_c, we get a one-sided power spectrum. We derived (33) using the Wiener-Khinchine theorem. Fig.11 shows the terms of a complex correlation function (38) for a one-sided power spectrum $\Gamma(f) = 0.5[1+\mathrm{sgn}(f)]\exp(-cf^2)$.

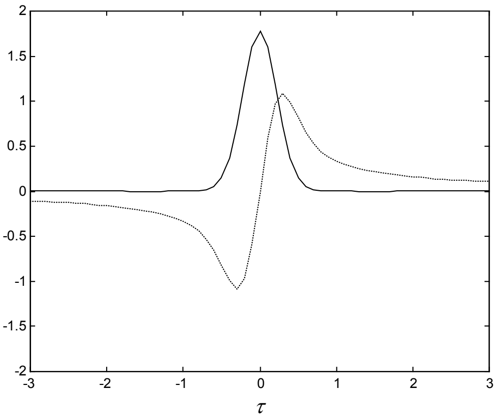


Fig.11. The real part (solid line) and the imaginary part (dotted line) of the complex correlation function for the power spectrum $\Gamma(f) = 0.5[1+\mathrm{sgn}(f)]\exp(-cf^2)$.

4.2 *Energy relations of the* WD *of 2-D complex signals*

Let us extend the results of the previous section for WD' s of 2-D complex signals. Let us recall that the real signal u(x_1, x_2) is represented by a set of two complex signals $\psi_1(x_1, x_2)$ and $\psi_3(x_1, x_2)$ which may have different energies. The subscripts q = 1 and q = 3 denote the corresponding quadrants of the ***f*** space. Only for certain signals both energies are equal as for example for all separable signals. The polar notation of the above signals of the form

$$\psi_1(x_1, x_2) = A_1(x_1, x_2)\, e^{j\Phi_1(x_1,x_2)} \quad ; \qquad \psi_3(x_1, x_2) = A_2(x_1, x_2)\, e^{j\Phi_2(x_1,x_2)} \tag{40}$$

defines two different local amplitudes

$$A_1^2(x_1,x_2) = u^2 + v^2 + v_1^2 + v_2^2 - 2(uv - v_1 v_2) \tag{41}$$

$$A_2^2(x_1,x_2) = u^2 + v^2 + v_1^2 + v_2^2 + 2(uv - v_1 v_2) \tag{42}$$

and two local phase functions $\Phi_1(x_1, x_2)$ and $\Phi_2(x_1, x_2)$. Here $v(x_1, x_2)$ is the 2-D complete Hilbert transform of u and $v_1(x_1, x_2)$ and $v_2(x_1, x_2)$ are partial Hilbert transforms in respect to the variables x_1 and x_2 [7], [9]. The energies of the signals u, v, v_1 and v_2 are equal, that is,

$$E_u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2(x_1,x_2)\,dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v^2(x_1,x_2)\,dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_1^2(x_1,x_2)\,dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_2^2(x_1,x_2)\,dx_1 dx_2 \tag{43}$$

Let us define the notion of the mutual energy given by the integral

$$E_m = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (uv - v_1 v_2)\,dx_1 dx_2 \tag{44}$$

However, it may be shown [9], that

$$E_{uv} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv\,dx_1 dx_2 = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_1 v_2\,dx_1 dx_2 \tag{45}$$

Therefore, $E_m = 2E_{uv}$. The energies of the complex signals ψ_1 and ψ_3 are

$$E_{\psi_1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_1 \psi_1^* dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1^2 dx_1 dx_2 = 4E_u - 4E_{uv} \tag{46}$$

$$E_{\psi_3} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_3 \psi_3^* dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_2^2 dx_1 dx_2 = 4E_u + 4E_{uv} \tag{47}$$

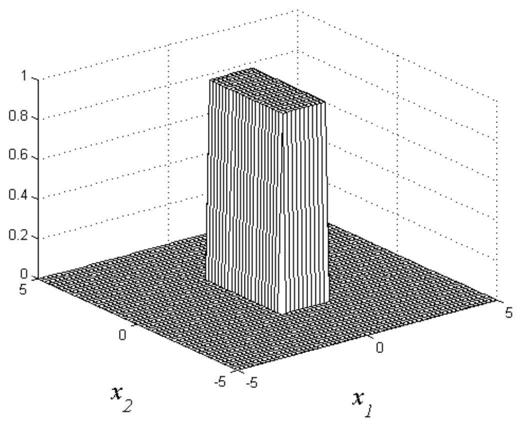


Fig.12. The signal $u(x_1, x_2)$ in the form of a cuboid.

Let us introduce a quantity called **N**ormalized **E**nergy **D**ifference (NED). It is defined by the formula

$$\text{NED} = \frac{E_{\psi_3} - E_{\psi_1}}{E_{\psi_3} + E_{\psi_1}} \tag{48}$$

A good insight to the energy relations yields a rotation the coordinate system. Since the rotation cancels the separability, the energy of the two signals given by (46) and (47) differs and NED is a function of the angle of rotation. Consider the 2-D analytic signal with the the real part u(x_1, x_2) in the form of the cuboidal signal of Fig.12.

Fig.13(left) shows an example of the dependence of the value of NED for this signal as a function of the angle of rotation γ. The parameter b/a equals the ratio of the sides of the rectangular support of the cuboid. Notice, that for long and narrow objects (large value of b/a) a small rotation gives a large energy difference. On the other hand for the square (b = a) there is no energy difference for all γ. Fig.13(right) shows the value of NED for the Gaussian signal defined by the equation of example 3 in section 3.1.2. The value of k^2 corresponds to the value of b/a of the cuboid. We observe, that the Gaussian signal is more sensitive to rotation in respect to the cuboid.

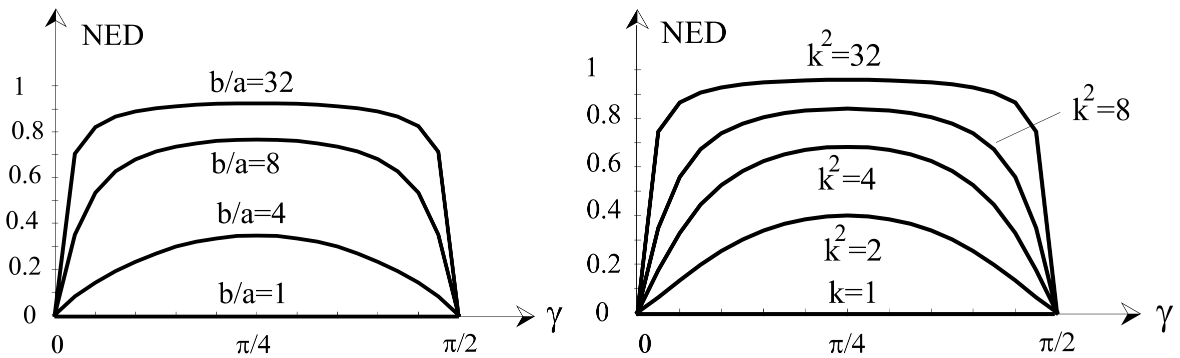


Fig.13. The dependence of the value of NED on the angle of rotation γ. Left for the cuboidal signal with different ratios b/a. Right for the Gaussian signal given by the Eq.(14).

Let us recall that the Gaussian signal has the smallest product

$$\text{Effective Support of the Signal}\times\text{Effective Support of the Spectrum} \tag{49}$$

in comparison to other signals including the cuboid. This causes a stronger sensitivity to rotation. The above examples presented the effect of rotation of the coordinate system on the distribution of the energies E_{ψ_1} and E_{ψ_3} for simple signal models given by mathematical functions. Let us have examples with natural images. Fig.14a shows the popular Lena image. We used a circular frame to avoid the effects caused by the rotation of the frame. Fig. 14b shows the change of the two energies caused by the rotation of the ***x*** coordinate system. Fig.15 shows the same effect for a more complicated computer generated image. As expected, the effect of rotation is smaller in comparison to the simpler image of Fig.14a.

Fig.14a. The Lena image in a cirsular frame.

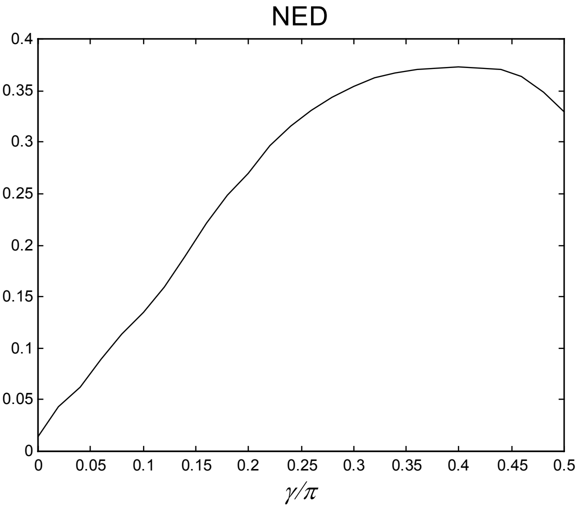


Fig.14b. The NED of the Lena image as a function of the rotation angle γ/

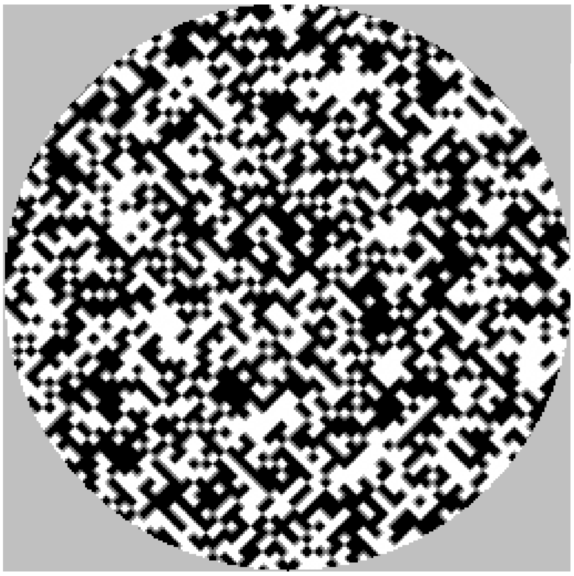


Fig.15.a. A computer generated test signal in a circular frame.

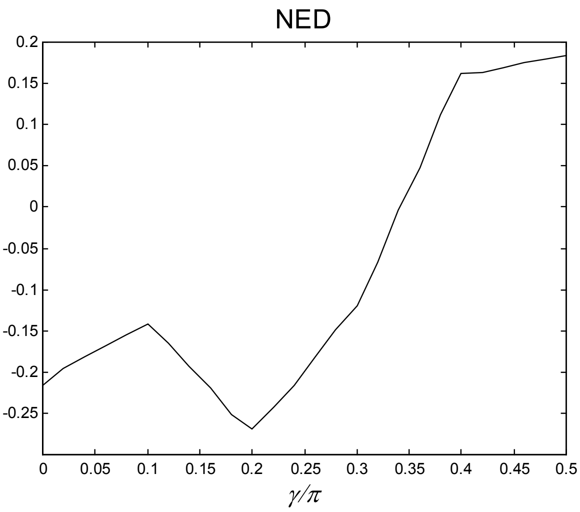


Fig.15b. The NED of the test signal as a function of the angle of rotation γ.

4.3 ***Moments for 2-D complex signals***

As in the 1-D case (see Eqs.(26) and (31)) let us define two-dimensional frequency domain moments $m_{kl}(x_1, x_2)$ and two-dimensional $\boldsymbol{x}$ domain moments $M_{kl}(f_1, f_2)$. The frequency domain moments are defined by the integral

$$m_{kl}(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (2\pi f_1)^k (2\pi f_2)^l \, W_q(x_1, x_2, f_1, f_2)\, df_1 df_2 \quad ; \quad q = 1\,,3 \tag{50}$$

where W_1 and W_3 are the distributions defined by the signals ψ_1 and ψ_3 respectively.The integration yields (derivation is similar as given in the Appendix)

$$\mathrm{m}_{\mathrm{k,l}}\left(x_1,x_2\right)=\left(\text{-j}\right)^{\mathrm{k}}\left(\text{-j}\right)^{\mathrm{l}}\lim_{\chi_1\to 0,\chi_2\to 0}\left\{\frac{\partial^{\mathrm{kl}}}{\partial^{\mathrm{k}}\chi_1\;\partial^{\mathrm{l}}\chi_2}\left[\psi_q\left(x_1+\chi_1/2,x_2+\chi_2/2\right)\psi_q^*\left(x_1-\chi_1/2,x_2-\chi_2/2\right)\right]\right\}$$

(51)

(q = 1, 3). For example the moment indexed (0,0) equals the local power of the signal $\psi_1(x_1, x_2)$

$$\mathrm{m}_{0,\,0}(x_1,x_2)=\psi_1(x_1,x_2)\psi_1^* x_1,x_2)=\mathrm{A}_1^2(x_1,x_2) \quad (52)$$

The next moments are

$$\mathrm{m}_{1,0}\left(x_1,x_2\right)=-\frac{\mathrm{j}}{2}\left[\frac{\partial\psi_{\mathrm{q}}}{\partial x_1}\psi_q^*+\frac{\partial\psi_{\mathrm{q}}^*}{\partial x_1}\psi_{\mathrm{q}}\right]\quad ;\ \mathrm{q}=1,3 \quad (53)$$

$$\mathrm{m}_{0,1}\left(x_1,x_2\right)=-\frac{\mathrm{j}}{2}\left[\frac{\partial\psi_{\mathrm{q}}}{\partial x_2}\psi_{\mathrm{q}}^*+\frac{\partial\psi_{\mathrm{q}}^*}{\partial x_2}\psi_{\mathrm{q}}\right]\quad ;\ \mathrm{q}=1,3 \quad (54)$$

The analytic signal ψ_n defines two partial complex frequencies [7] [9], [10] of the form

$$s_{1x_1}(x_1,x_2)=\frac{\partial}{\partial x_1}\mathrm{Log}\left[\psi_{\mathrm{q}}(x_1,x_2)\right]=\alpha_{1x_1}+\mathrm{j}\omega_{1x_1} \quad (55)$$

$$s_{1x_2}(x_1,x_2)=\frac{\partial}{\partial x_2}\mathrm{Log}\left[\psi_{\mathrm{q}}(x_1,x_2)\right]=\alpha_{1x_2}+j\omega_{1x_2} \quad (56)$$

These local frequencies can be written in the terms of the moments

$$\alpha_{1x_1}=\frac{0.5\frac{\partial}{\partial x_1}\mathrm{m}_{0,0}}{\mathrm{m}_{0,0}}\qquad ;\qquad \omega_{1x_1}=\frac{\mathrm{m}_{1,0}}{\mathrm{m}_{0,0}} \quad (57)$$

$$\alpha_{1x_2}=\frac{0.5\frac{\partial}{\partial x_2}m_{0,0}}{m_{0,0}}\qquad ;\qquad \omega_{1x_2}=\frac{\mathrm{m}_{0,1}}{\mathrm{m}_{0,0}} \quad (58)$$

The ***x*** domain moments are defined by the integral

$$\mathrm{M}_{\mathrm{k,l}}(f_1,f_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1^k x_2^l \mathrm{W}_{\mathrm{q}}\left(x_1,x_2,f_1,f_2\right)\mathrm{d}x_1\mathrm{d}x_2\quad ;\quad \mathrm{q}=1,3 \quad (59)$$

The integration yields (derivation is similar as given in the Appendix)

$$\mathrm{M}_{\mathrm{k,l}}(f_1,f_2)=\left(\frac{\mathrm{j}}{2\pi}\right)^{\mathrm{k}}\left(\frac{\mathrm{j}}{2\pi}\right)^{\mathrm{l}}\lim_{\mu_1\to 0,\mu_2\to 0}\left\{\frac{\partial^{kl}}{\partial\mu_1^k\partial\mu_2^l}\left[\Gamma_n\left(f_1+\mu_1/2,f_2+\mu_2/2\right)\Gamma_n^*\left(f_1-\mu_1/2,f_2-\mu_2/2\right)\right]\right\} \quad (60)$$

Especially, the moment $\mathrm{M}_{0,0}(f_1,f_2)$ for q = 1 yields a single-quadrant power spectrum in the first quadrant

$$\mathrm{M}_{0,0}(f_1,f_2)=\Gamma_1(f_1,f_2)\Gamma_1^*(f_1,f_2)=16\times\mathbf{1}(f_1,f_2)\,|\mathrm{U}(\mathrm{f}_1,\mathrm{f}_2)|^2 \quad (61)$$

Analogously, for q = 3 we get a single-quadrant power spectrum in the third quadrant. The next moment $\mathrm{M}_{1,0}$ is

$$\mathrm{M}_{1,0}\left(f_1,f_2\right)=\frac{\mathrm{j}}{4\pi}\left\{\frac{\partial\Gamma_q(f_1,f_2)}{\partial f_1}\Gamma_{\mathrm{q}}^*(f_1,f_2)-\frac{\partial\Gamma_{\mathrm{q}}^*(f_1,f_2)}{\partial f_1}\Gamma_q(f_1,f_2)\right\} \quad (62)$$

The moment $\mathrm{M}_{0,1}$ is given by the same formula changing $\partial/\partial\mathrm{x}_1$ by $\partial/\partial\mathrm{x}_2$.

4.4 ***The 2-D complex correlation function***

Similarly, as in the 1-D case, let us derive the power spectrum (60) as the inverse Fourier transform of a complex 2-D correlation function with a single-quadrant spectrum. We start with the frequency domain marginal of the 4-D WVD given by the integral

$$\overline{W_q(x_1,x_2,f_1,f_2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{W_q}\left(x_1,x_2,f_1,f_2\right)\mathrm{d}x_1\mathrm{d}x_2 \tag{63}$$

(q = 1, 3). Let us insert the distribution W given by the Eq.(10). By changing the integration order we get

$$\overline{\mathrm{W_q}(x_1,x_2,f_1,f_2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_\mathrm{q}(\alpha_+)\psi_q^*(\alpha_-)\mathrm{d}x_1\mathrm{d}x_2\right\}\mathrm{e}^{-\mathrm{j}2\pi(f_1\chi_1+f_2\chi_2)}\mathrm{d}\chi_1\mathrm{d}\chi_2$$

(64)

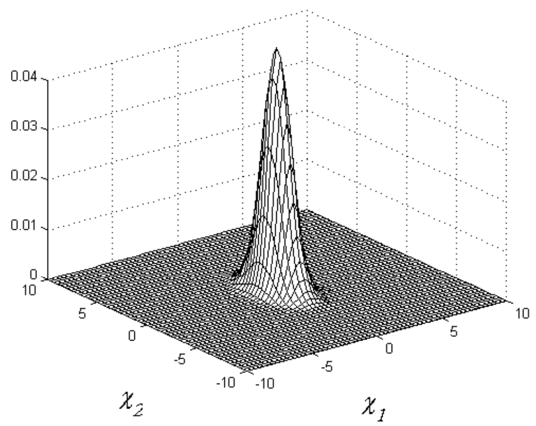
The insertion of the analytic signal [7], [8]

$$\psi_1(x_1, x_2) = \mathrm{u}(x_1, x_2) - \mathrm{v}(x_1, x_2) + \mathrm{j}\,[\mathrm{v}_1(x_1, x_2) + \mathrm{v}_2(x_1, x_2)] \tag{65}$$

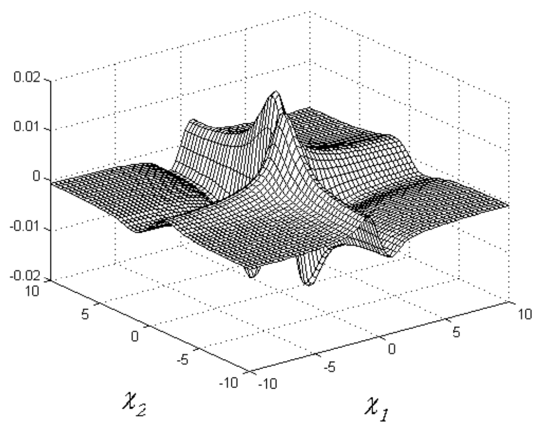
where $\mathrm{v} = \mathrm{H(u)}$ is the total Hilbert transform of u and $\mathrm{v}_1 = \mathrm{H}_1(\mathrm{u})$ and $\mathrm{v}_2 = \mathrm{H}_2(\mathrm{u})$ are partial Hilbert transforms. The evaluation integral in parenthesis yields an analytic correlation function of the same structure as the signal ψ_1 of the form

$$\rho_{\mathrm{c1}}(\chi_1,\chi_2) = 4\left[\rho_{\mathrm{uu}}\left(\chi_1,\chi_2\right)-\rho_{\mathrm{vv}}\left(\chi_1,\chi_2\right)\right]+\mathrm{j}4\left[\rho_{\mathrm{v_1v_1}}\left(\chi_1,\chi_2\right)+\rho_{\mathrm{v_2v_2}}\left(\chi_1,\chi_2\right)\right] \tag{66}$$

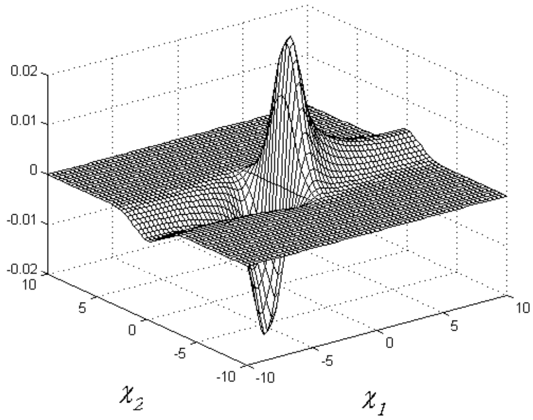
This correlation function is analytic and its Fourier transform yields a single quadrant power spectrum (see Eq.(61)). We derived the Wiener-Khinchine theorem for complex signals with single-quadrant spectra. All the above quantities have been derived starting with complex signals $\psi_1(x_1, x_2)$. A similar derivation can be applied to the analyic signal $\psi_3(x_1, x_2)$ yielding an analytic correlation function corresponding to a third quadrant power spectrum. Therefore, we have two sets of moments, two correlation functions, two single-quadrant power spectra (in the first and third quadrant respectively) and two sets of partial local complex frequencies given by the equations (57) and (58). Fig.16 shows the real parts and imaginary parts of the 2-D complex correlations functions for a 2-D separabl*e* Gaussian signal.



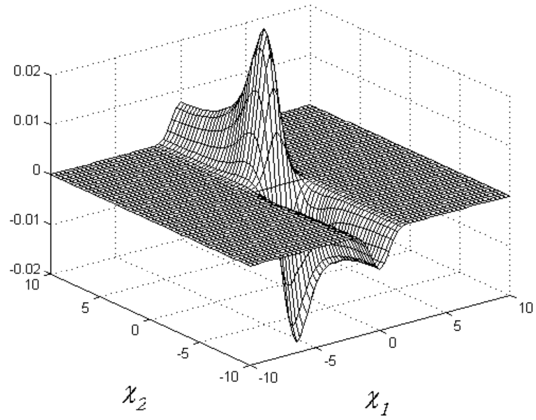
a) $\rho_{\mathrm{uu}}(\chi_1, \chi_2)$



b) $\rho_{\mathrm{vv}}(\chi_1, \chi_2) = \mathrm{H}[\rho_{\mathrm{uu}}(\chi_1, \chi_2)]$



c) $\rho_{v1v1}(\chi_1, \chi_2) = H_1[\rho_{uu}(\chi_1, \chi_2)]$



c) $\rho_{v2v2}(\chi_1, \chi_2) = H_2[\rho_{uu}(\chi_1, \chi_2)]$

Fig.16. The four terms of the analytic correlation function given by the Eq.(64) corresponding to a Gaussian real signal $u(x_1, x_2)$

4.5 ***Energy relations and moments for* WD'*s of 3-D complex signals***

The extension of the energy relations and the derivation of moments presented in the previous sections for 1-D and 2-D signals for 3-D complex signals is straightforward. A 3-D real signal is represented by a set of four complex signals with a single octant spectra of the form [7], [9]

$$\psi_1(x_1,x_2,x_3) = \psi_8^*(x_1,x_2,x_3) = \mathrm{u}\text{-}\mathrm{v}_{12}\text{-}\mathrm{v}_{13}\text{-}\mathrm{v}_{23}+\mathrm{j}[\mathrm{v}_1+\mathrm{v}_2+\mathrm{v}_3\text{-}\mathrm{v}] = \mathrm{A}_1\mathrm{e}^{\mathrm{j}\Phi_1} \tag{67}$$

$$\psi_2(x_1,x_2,x_3) = \psi_7^*(x_1,x_2,x_3) = \mathrm{u}+\mathrm{v}_{12}+\mathrm{v}_{13}\text{-}\mathrm{v}_{23}+\mathrm{j}[\text{-}\mathrm{v}_1+\mathrm{v}_2+\mathrm{v}_3+\mathrm{v}] = \mathrm{A}_2\mathrm{e}^{\mathrm{j}\Phi_2} \tag{68}$$

$$\psi_3(x_1,x_2,x_3) = \psi_6^*(x_1,x_2,x_3) = \mathrm{u}+\mathrm{v}_{12}\text{-}\mathrm{v}_{13}+\mathrm{v}_{23}+\mathrm{j}[\mathrm{v}_1\text{-}\mathrm{v}_2+\mathrm{v}_3+\mathrm{v}] = \mathrm{A}_3\mathrm{e}^{\mathrm{j}\Phi_3} \tag{69}$$

$$\psi_4(x_1,x_2,x_3) = \psi_5^*(x_1,x_2,x_3) = \mathrm{u}\text{-}\mathrm{v}_{12}+\mathrm{v}_{13}+\mathrm{v}_{23}+\mathrm{j}[\text{-}\mathrm{v}_1\text{-}\mathrm{v}_2+\mathrm{v}_3+\mathrm{v}] = \mathrm{A}_4\mathrm{e}^{\mathrm{j}\Phi_4} \tag{70}$$

where v is the complete Hilbert transform in respect to all three variables, v_{12}, v_{13} and v_{23} are partial Hilbert transforms in respect to two variables and v_1, v_2 and v_3 in respect to a single variable. The real signal $u(x_1, x_2, x_3)$ can be reconstructed in terms of the complex signals using the Eq.(4). The energies of these signals are given by the integral

$$\mathrm{E}_{\psi_q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_q\psi_q^*\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{A}_q^2\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\ ;\ \mathrm{q} = 1, 2, 3, 4 \tag{71}$$

The energies of the four signals are different except the case of signals with a full symmetry of the spectrum including separable signals. For separable signals all the amplitudes are equal and given by the formula

$$\mathrm{A}_0^2 = \mathrm{u}^2+\mathrm{v}^2+\mathrm{v}_1^2+\mathrm{v}_2^2+\mathrm{v}_{12}^2+\mathrm{v}_{13}^2+\mathrm{v}_{23}^2 \tag{72}$$

4.5.1 ***Relativity of the 3-D Fourier spectrum***

As in the 2-D case the Fourier spectrum of a 3-D signals is relative and may change with the shift or rotation of the 3-D Cartesian signal coordinates. Let us study the effect of rotation. Let us recall the Parseval's equality for 3-D real signals

$$E=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u^2(x_1,x_2,x_3)dx_1dx_2dx_3=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U^2(f_1,f_2,f_3)df_1df_2df_3 \qquad (73)$$

The value of E defines the total energy of the real signals. The integration in the frequency domain yields a sum of the energies of succesive eight octants. For a certain class of signals, especially separable signals, all eight octants have the same energy equal E/8. The rotation of the coordinate system can change the energy distribution between the octants. In consequence the energies of the four complex signals (67) to (70) may differ. Let us illustrate this change using a separable 3-D Gaussian signal. The rotation of the 3-D Cartesian coordinate system is defined by three Euler's angles denoted δ, ψ and φ. Fig.17 shows the dependence of the energy distribution between four octants enforced by the simultaneous change of all three angles, only two angles or a single angle. The parameter k defines the ellipticity of cross-sections of the 3-D Gaussian signal. Notice, that due to the Hermitian symmetry it is sufficient to observe the changes for four octants.

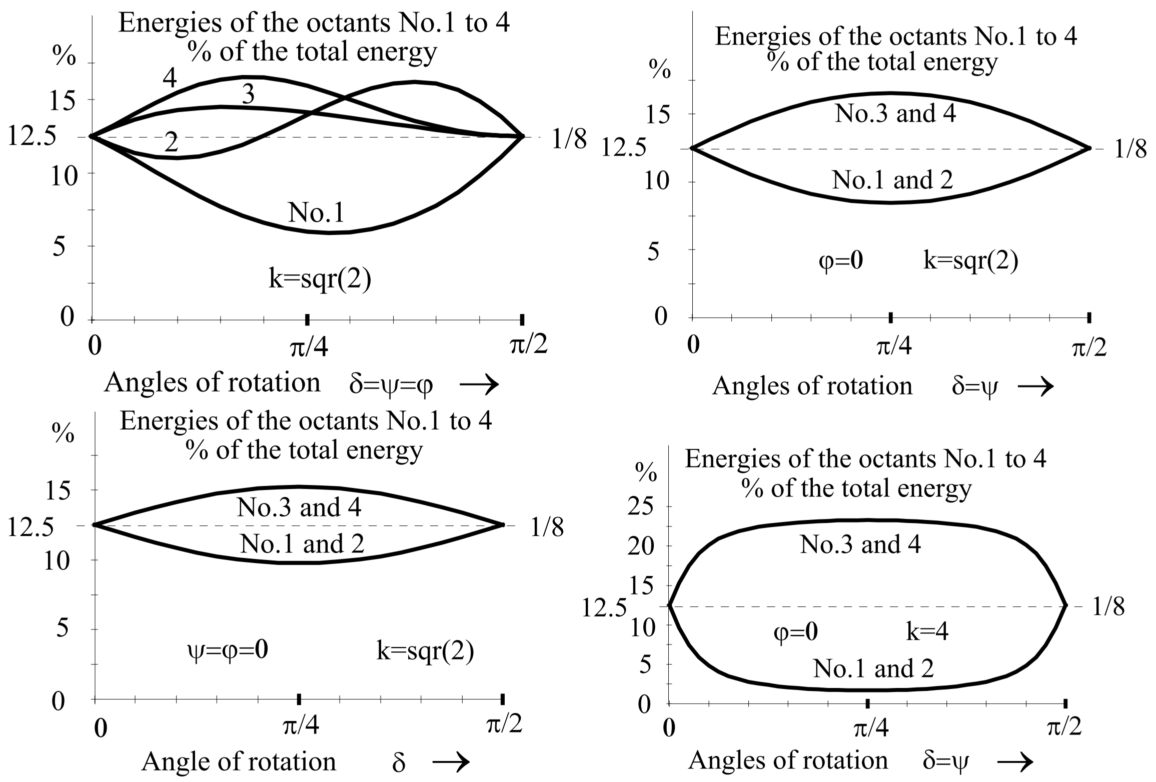


Fig.18. The rotation of a 3-D Gaussian signal. The dependence of the energy in octants on the angles of rotation.

4.6 *Moments for 3-D complex signals*

The four WD's denoted $W_q(x_1, x_2, x_3, f_1, f_2, f_3)$ (q = 1, 2, 3, 4) define each ***f*** domain moments and ***x*** domain moments. The ***f*** domain moments are defined by the formula

$$m_{klm}(x_1,x_2,x_3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(2\pi f_1)^k(2\pi f_2)^l(2\pi f_3)^m\,W_q(x_1,x_2,x_3,f_1,f_2,f_3)df_1df_2df_3 \qquad (74)$$

and the derivation yields (we resigned to label m_{klm} with the subscript q to avoid to many subscripts).

$\mathrm{m}_{\mathrm{klm}}(x_1, x_2, x_3) =$ (75)

$$(\text{-j})^{\mathrm{k}}(\text{-j})^{\mathrm{l}}(\text{-j})^{\mathrm{m}} \lim_{\text{all }\chi\to 0} \frac{\partial^{\mathrm{klm}}}{\partial^{\mathrm{k}}\partial^{\mathrm{l}}\partial^{\mathrm{m}}}\left\{\psi_q\left(x_1+\frac{\chi_1}{2}, x_2+\frac{\chi_2}{2}, x_3+\frac{\chi_3}{3}\right)\psi_{\mathrm{q}}^{*}\left(x_1-\frac{\chi_1}{2}, x_2-\frac{\chi_2}{2}, x_3-\frac{\chi_3}{3}\right)\right\}$$

The derivation is similar as given in the Appendix. The moment indexed (0,0,0) (***x*** domain marginal) equals the local power of the signal

$$\mathrm{m}_{\mathrm{ooo}}(x_1, x_2, x_3) = \psi_{\mathrm{q}}(x_1, x_2, x_3)\ \ \psi_{\mathrm{q}}^{*}(x_1, x_2, x_3) = \mathrm{A}_{\mathrm{q}}^{2}(x_1, x_2, x_3) \tag{76}$$

The ***f*** domain moments are defined by the integral

$$\mathrm{M}_{\mathrm{klm}}(f_1, f_2, f_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^{\mathrm{k}} x_2^{\mathrm{l}} x_3^{\mathrm{m}} \mathrm{W}_{\mathrm{q}}\left(x_1, x_2, x_3, f_1, f_2, f_3\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \tag{77}$$

and the derivation similar as given in the Appendix yields

$\mathrm{M}_{\mathrm{klm}}(f_1, f_2, f_3) =$ (78)

$$\left(\frac{\mathrm{j}}{2\pi}\right)^{\mathrm{k}}\left(\frac{\mathrm{j}}{2\pi}\right)^{\mathrm{l}}\left(\frac{\mathrm{j}}{2\pi}\right)^{\mathrm{m}} \lim_{\text{all }\mu\to 0} \frac{\partial^{\mathrm{klm}}}{\partial^{\mathrm{k}}\partial^{\mathrm{l}}\partial^{\mathrm{m}}}\left\{\Gamma_{\mathrm{q}}\left(f_1+\frac{\mu_1}{2}, f_2+\frac{\mu_2}{2}, f_3+\frac{\mu_3}{3}\right)\Gamma_{\mathrm{q}}^{*}\left(f_1-\frac{\mu_1}{2}, f_2-\frac{\mu_2}{2}, f_3-\frac{\mu_3}{3}\right)\right\}$$

Analogously to the Eqs.(38) and (66) a 3-D complex correlation function can be derived using the ***x*** domain mean value of the WVD. As well, the local partial complex frequencies may be represented using appropriate moments.

5 Conclusion

In this paper we presented the extension of the notion of the Wigner-Ville time-frequency distribution of 1-D analytic signals for 2-D and 3-D complex signals with single orthant spectra. Especially, 2-D signals with single-quadrant spectra define two different 4-D WD and 3-D signals with single-octant spectra define four different 6-D WD. The fundamental question is, whether it is reasonable to apply complex signals instead of real signals. Somebody could argument that using real signals we operate with a single distribution instead of two, four and so on. However, using real signals the distribution is defined using the full Fourier frequency space while the complete information about the signal is contained in the half Fourier frequency space. We produce also much more cross terms in comparison to cross terms produced using complex signals. On the other hand, the necessity of calculation a set of two distributions in the 2-D case or four in the 3-D case cannot be classified only as an inconvenience. Firstly, we have no choice, because it is enforced by the nature of the complex notation of Fourier transformations. The half Fourier frequency space is divided into two quadrants or four octants which in general contain different information about the signal and define signals with different energies. Secondly, in digital signal processing, using the frequency domain algorithm of calculation of the WD the number of operations used to calculate these two (or four) distributions is smaller, than should be used to calculate the single distribution defined by a real signal.

Thirdly, the two different distributions can fascillitate the evaluation of the informations contained in the distributions. The development of the presented extensions of the notion of the Wiegner-Ville distribution could not be done without the use of the notion of complex signals with single orthant spectra given in the previous paper of Hahn [7]. We forecast, that the presented theory has many potential applications and hope that our paper can be used to start next research in this field. Especially, better methods of graphical representations of four and six-dimensional functions should be applied. The presented extensions of the Wiener-Khinschine theorem and the definitions of moments were also possible using the complex notation. Let us mention, that many details of the presented theory have been developped during the preparation of the manuscript. In this paper we used the theory of signals with single-orthant spectra to show, that the complete spectral description of a n-D signal is carried by the half Fourier space. This fact is well described by many authors for 1-D signals. The extension to higher dimensions is obvious. However, we have not found a clear statement about it in available sources. The theoretical part of this paper including the derivation of the formulae has been done by S.Hahn. Dr.A.Buchowicz checked these derivations. The computational part has been developped by both authors in a close cooperation.

Acknowledgement

This work was supported by a grant of KBN (Committee for Scientific Research, Poland).

Historical remark

This report is a copy with few improvements of a paper submitted on 2 December 1997 for possible publication in the IEEE Transactions on Signal Processing. However, the Associate Editor, Dr.Mohamood R.Azimi-Sadjadi refused to accept the paper. Recently we read our paper again and observed that some important itoms, especially the extension of the Wiener-Khinchine theorem for n-D analytic signals has never been described in any paper we have been able to read. In consequence, we decided to write this document as an official report of the Institute of Radioelectronics, Warsaw University of Technology.

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Appendix : The derivation of the moments defined by the WVD

We shall use the following formulae: The Fourier transform and inverse Fourier transform of $(j2\pi f)^k$ of the form

$$\int_{-\infty}^{\infty} \left(j2\pi f\right)^k e^{-j2\pi f y} df = \left(-1\right)^k \delta^{(k)}(y) \qquad (A1)$$

$$\int_{-\infty}^{\infty} \left(j2\pi f\right)^k e^{j2\pi f y} df = \delta^{(k)}(y) \qquad (A2)$$

where $\delta^{(k)}(y)$ is the k-th derivative of the delta distribution (generalized function). For any function f(y) differentiable k-times at the origin the following relation holds (see [8],page 14)

$$\int_{-\infty}^{\infty} f(y)\delta^{(k)}(y)dy = (-1)^k f^{(k)}(0) \qquad (A3)$$

The time dependent moment $m_k(t)$ is defined by the integral

$$m_k(t) = \int_{-\infty}^{\infty} (2\pi f)^k W\left(t, f\right) df \qquad (A4)$$

where k = 0, 1, 2... is the order of the moment. The multiplication and division by j^k and insertion of W(t, f) given by (7) yields

$$m_k(t) = (-j)^k \int_{-\infty}^{\infty} \psi\left(t+\tau/2\right)\psi^*(t-\tau/2)\left\{\int_{-\infty}^{\infty}\left(j2\pi f\right)^k e^{-j2\pi f\tau} df\right\} d\tau \qquad (A5)$$

The intergal in paranthesis {} yields $(-1)^k\,\delta^{(k)}(\tau)$ (see (A1)) and due to (A3) we get

$$m_k(t) = \left(-j\right)^k \lim_{\tau\to 0}\left\{\frac{d^k}{dt^k}\left[\psi\left(t+\tau/2\right)\psi^*\left(t-\tau/2\right)\right]\right\} \qquad (A6)$$

The application of the operator "lim" is necessary, since we should first differentiate and then insert $\tau = 0$.

Notice that the product of $(-1)^k\,(-1)^k$ from (A1) and (A3) equals 1.

The frequency dependent moment $M_k(f)$ is defined by the intergal

$$M_k(f) = \int_{-\infty}^{\infty} t^k W\left(t, f\right) dt \qquad (A7)$$

The multiplication and division by $(j2\pi)^k$ and insertion of W(t, f) given by (8) yields

$$M_k(f) = \left(-\frac{j}{2\pi}\right)^k \int_{-\infty}^{\infty} \Gamma\left(f+\mu/2\right)\Gamma^*(f-\mu/2)\left\{\int_{-\infty}^{\infty}\left(j2\pi t\right)^k e^{j2\pi\mu t} dt\right\} d\mu \qquad (A8)$$

The intergal in paranthesis {} yields $\delta^{(k)}(\mu)$ (see (A2)) and due to (A3) we get

$$\mathrm{M_k}(f) = \left(\frac{\mathrm{j}}{2\pi}\right)^{\mathrm{n}} \lim_{\mu\to 0}\left\{\frac{\mathrm{d^k}}{\mathrm{d}\mu^{\mathrm{k}}}\left[\Gamma\left(f+\mu/2\right)\Gamma^{*}\left(f-\mu/2\right)\right]\right\} \qquad \text{(A9)}$$

Notice that the product of $(\text{-j})^{\mathrm{k}}\,(\text{-}1)^{\mathrm{k}}$ (see (A3)) equals j^{k}.

The application of the same procedure, as used above enables the derivation of the moments defined by the 4-D WD, that is $\mathrm{m_{kl}}(x_1, x_2)$ and $\mathrm{M_{kl}}(f_1, f_2)$ defined by the Eqs.(49) and (58). Due to the separability of 2-D delta distribution the 1-D formulae (A1) and (A2) can be applied. The formula (A3) takes the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{f}\left(y_1, y_2\right)\delta^{(\mathrm{k})}(y_1)\delta^{(\mathrm{l})}(y_2)\mathrm{d}y_1\mathrm{d}y_2 = (\text{-}1)^{\mathrm{k}}\left(\text{-}1\right)^{\mathrm{l}} f^{(\mathrm{kl})}(0,0) \qquad \text{(A10)}$$

Since the dervation is similar,as in the 1-D case, details are not presented here.