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The analytic, quaternionic and monogenic 2-D complex delta distributions

by

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1. Introduction

The notion of an n-D analytic signal with single-quadrant spectrum has been defined by the author in 1992 [1]. The description can be found also in [2]. Recently Bülow and Sommer defined the notion of a quaternionic 2-D complex signal [3], [4] and Felsberg and Sommer defined the notion of a monogenic 2-D signal [5], [6]. Here we show, that all three signals, that is the analytic, quaternionic and monogenic signals have a common representation in the form of a convolution of a real signal $u(x_1, x_2)$ with an appropriate complex delta distribution, which is different for the above mentioned three signals.

2. The analytic 2-D signal

Consider a real signal $u(\boldsymbol{x})$ is defined in the Cartesian signal plane $\boldsymbol{x} = (x_1, x_2)$ and its Fourier spectrum

$$U(\boldsymbol{f}) = F[u(x_1, x_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1, x_2)\exp[-j2\pi(x_1 f_1 + x_2 f_2)]dx_1 dx_2 \tag{1}$$

defined in the Fourier frequency plane $\boldsymbol{f} = (f_1, f_2)$. This plane is a sum of four quadrants. However, due to the Hermitian symmetry of $U(\boldsymbol{f})$, [$U(f_1, f_2) = U^*(-f_1, -f_2)$ and $U(f_1, -f_2) = U^*(-f_1, f_2)$], all spectral information is contained in a half-plane , for example the half-plane $f_1 > 0$. In consequence in general a given real signal is represented by two analytic signals with single quadrant spectra. For example the first is given by the inverse Fourier transform of $U(\boldsymbol{f})$ multiplied by the frequency domain operator

$$\Delta_1(\boldsymbol{f}) = [1+\mathrm{sgn}(f_1)]\,[1+\mathrm{sgn}(f_2)] = [1+\mathrm{sgn}(f_1) + \mathrm{sgn}(f_2) + \mathrm{sgn}(f_1)\mathrm{sgn}(f_2)] \tag{2}$$

and the second by the multiplication by

$$\Delta_3(\boldsymbol{f}) = [1+\mathrm{sgn}(f_1)]\,[1-\mathrm{sgn}(f_2)] = [1+\mathrm{sgn}(f_1) - \mathrm{sgn}(f_2) - \mathrm{sgn}(f_1)\mathrm{sgn}(f_2)] \tag{3}$$

Since multiplication in the frequency domain corresponds to the convolution in the signal domain, the first signal is given by the formula

$$\psi_1(\boldsymbol{x}) = F^{-1}[\Delta_1(\boldsymbol{f}) ** u(x_1, x_2)] = \psi_{\delta 1}(\boldsymbol{x}) ** u(x_1, x_2) \tag{4}$$

and the second (subscript 3 represents the accepted labeling of quadrants)

$$\psi_3(\boldsymbol{x}) = F^{-1}[\Delta_3(\boldsymbol{f}) ** u(x_1, x_2)] = \psi_{\delta 3}(\boldsymbol{x}) ** u(x_1, x_2) \tag{5}$$

The signals $\psi_{\delta 1}(\boldsymbol{x})$ and $\psi_{\delta 3}(\boldsymbol{x})$ are called complex delta distributions [7]. Therefore, the analytic signals can be redefined as convolutions of a real signal with complex delta distributions. The inverse Fourier transform of (2) yields

$$\psi_{\delta 1}(x_1, x_2) = \delta(x_1, x_2) - \frac{1}{\pi^2 x_1 x_2} + j\left[\frac{\delta(x_2)}{\pi x_1} + \frac{\delta(x_1)}{\pi x_2}\right] \tag{6}$$

and analogously

$$\psi_{\delta 3}(x_1, x_2) = \delta(x_1, x_2) + \frac{1}{\pi^2 x_1 x_2} + j\left[\frac{\delta(x_2)}{\pi x_1} - \frac{\delta(x_1)}{\pi x_2}\right] \tag{7}$$

The insertion of (6) in (4) yields the first analytic signal

$$\psi_1(\boldsymbol{x}) = u(\boldsymbol{x}) - v(\boldsymbol{x}) + j[v_1(\boldsymbol{x}) + v_2(\boldsymbol{x})] \tag{8}$$

and the insertion of (7) into (5) yields the second analytic signal

$$\psi_3(\boldsymbol{x}) = \mathrm{u}(\boldsymbol{x}) + \mathrm{v}(\boldsymbol{x}) + \mathrm{j}[\mathrm{v}_1(\boldsymbol{x}) - \mathrm{v}_2(\boldsymbol{x})] \tag{9}$$

where $\mathrm{v}(\boldsymbol{x})$ is the total Hilbert transform of $\mathrm{u}(\boldsymbol{x})$ w.r.t . both variables and $\mathrm{v}_1(\boldsymbol{x})$ and $\mathrm{v}_2(\boldsymbol{x})$ are partial Hilbert transforms w.r.t. a single variable (see Appendix 1 for integral forms).

3. The quaternionic complex 2-D signal

The notion of a quaternionic complex signal has been defined in [4] and [5] using the quaternionic Fourier transform of $\mathrm{u}(\boldsymbol{x})$ of the form

$$U_q(f_1, f_2) = \mathrm{F}_q[u(x_1, x_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1, x_2)\exp(-\,\mathrm{i}2\pi f_1 x_1)\exp(-\,\mathrm{j}2\pi f_2 x_2)\,\mathrm{d}x_1\mathrm{d}x_2 \tag{10}$$

where i and j, are imaginary units each equal sqr(-1). The inverse Fourier transform over the first quadrant, i.e.,

$$\psi_q(x_1, x_2) = \int_0^{\infty}\int_0^{\infty} \mathrm{U}_q(f_1, f_2)\ \exp(\mathrm{i}2\pi f_1 x_1)\ \exp(\mathrm{j}2\pi f_2 x_2)\ \mathrm{d}f_1\mathrm{d}f_2 \tag{11}$$

yields the quaternionic complex signal of the form

$$\psi_q(\boldsymbol{x}) = \mathrm{u}(\boldsymbol{x}) + \mathrm{i}\mathrm{v}_1(\boldsymbol{x}) + \mathrm{j}\mathrm{v}_2(\boldsymbol{x}) + \mathrm{k}\,\mathrm{v}(\boldsymbol{x}) \tag{12}$$

The notations are the same as in (8) and (9) and the imaginary units fulfill the following algebra $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}$, $\mathrm{ik} = -\mathrm{ki} = -\mathrm{j}$. The insertion in (12) $\mathrm{u}(\boldsymbol{x}) = \delta(x_1, x_2)$ yields the notion of the quaternionic complex delta distribution [8] of the form

$$\psi_{q\delta}(x_1, x_2) = \delta(x_1, x_2) + \mathrm{i}\frac{\delta(x_2)}{\pi x_1} + \mathrm{j}\frac{\delta(x_1)}{\pi x_2} + \mathrm{k}\frac{1}{\pi^2 x_1 x_2} \tag{13}$$

and the quaternionic complex signal can be alternatively defined by the convolution

$$\Psi_q(\boldsymbol{x}) = \mathrm{u}(\boldsymbol{x}) ** \psi_{q\delta}(\boldsymbol{x}) \tag{14}$$

The comparison of the properties of analytic and quaternionic complex signals can be found in [x].

4. The monogenic complex 2-D signal.

In recent papers [5], [6] the authors defined the notion of a monogenic complex signals. We do not intend here to describe the derivations which can be found in [5] and [6]. The monogenic complex signal has the form

$$\psi_M(\boldsymbol{x}) = \mathrm{u}(\boldsymbol{x}) - \mathrm{i}\theta_{r1}(\boldsymbol{x}) - \mathrm{j}\theta_{r2}(\boldsymbol{x}) \tag{15}$$

where $\theta_{r1}(\boldsymbol{x})$ and $\theta_{r2}(\boldsymbol{x})$ are the Riesz transforms of $\mathrm{u}(\boldsymbol{x})$ defined by the convolutions

$$\theta_{r1}(x_1, x_2) = \mathrm{u}(x_1, x_2) ** \frac{x_1}{2\pi\left(\sqrt{x_1^2 + x_2^2}\right)^3} \quad ; \quad \theta_{r2}(x_1, x_2) = \mathrm{u}(x_1, x_2) ** \frac{x_2}{2\pi\left(\sqrt{x_1^2 + x_2^2}\right)^3} \tag{16}$$

The Fourier transforms of the Riesz kernels are

$$\mathrm{F}\left[\frac{-x_1}{2\pi\left(\sqrt{x_1^2 + x_2^2}\right)^3}\right] = \frac{f_1}{\sqrt{f_1^2 + f_2^2}} \quad ; \quad \mathrm{F}\left[\frac{-x_2}{2\pi\left(\sqrt{x_1^2 + x_2^2}\right)^3}\right] = \frac{f_2}{\sqrt{f_1^2 + f_2^2}} \tag{17}$$

Therefore, the alternative definition of $\theta_{r1}(\boldsymbol{x})$ and $\theta_{r2}(\boldsymbol{x})$ is given by the inverse Fourier transforms

$$\theta_{r1}(x_1,x_2)=F^{-1}\left[\frac{f_1}{\sqrt{f_1^2+f_2^2}}U(f_1,f_2)\right]\quad;\quad\theta_{r2}(x_1,x_2)=F^{-1}\left[\frac{f_2}{\sqrt{f_1^2+f_2^2}}U(f_1,f_2)\right]\tag{18}$$

For the signal u($\boldsymbol{x}$) = δ(x_1, x_2) we define the monogenic complex delta distribution

$$\psi_{M\delta}(x_1,x_2)=\delta(x_1,x_2)-i\frac{x_1}{2\pi\left(\sqrt{x_1^2+x_2^2}\right)^3}-j\frac{x_2}{2\pi\left(\sqrt{x_1^2+x_2^2}\right)}\tag{19}$$

Analogously to the Eqs.(4), (5) and (14) the monogenic complex signal can be redefined as a convolution of the real signal with the monogenic complex delta distribution:

$$\Psi_M(\boldsymbol{x}) = u(\boldsymbol{x})** \psi_{M\delta}(\boldsymbol{x})\tag{20}$$

5. Auto-convolutions of the complex delta distributions

The auto-convolution of the analytic complex delta distribution given by the Eq.(6) is

$$\psi_{\delta 1}(\boldsymbol{x})** \psi_{\delta 1}(\boldsymbol{x}) = 4\psi_{\delta 1}(\boldsymbol{x})\tag{21}$$

The same autoconvolution for the quaternionic complex delta distribution given by (14) is

$$\psi_{q\delta}(\boldsymbol{x})** \psi_{q\delta}(\boldsymbol{x}) = 2\psi_{q\delta}(\boldsymbol{x})\tag{22}$$

The derivation of (21) and (22) is given in the Appendix 2. Notice, that the formulae (21) and (22) differ only by the numerical factor 4 and 2. The auto-convolution of the monogenic complex delta distribution is

$$\psi_{M\delta}(x_1,x_2)**\psi_{M\delta}(x_1,x_2)=2\delta(x_1,x_2)-i\frac{x_1}{\pi\left(\sqrt{x_1^2+x_2^2}\right)^3}-j\frac{x_2}{\pi\left(\sqrt{x_1^2+x_2^2}\right)^3}=2\psi_{M\delta}(x_1,x_2)\tag{23}$$

and remarkably has the same form, as (22). Derivation of (23) is given in the Appendix 3.

.6. Auto-convolutions of the complex signals

Let us denote by $\psi_*(\boldsymbol{x}) = \psi_1(\boldsymbol{x})**\psi_1(\boldsymbol{x})$ the auto-convolution of the analytic signal (4), and by $u_*(\boldsymbol{x})$ $=u(\boldsymbol{x})**u(\boldsymbol{x})$ the auto-convolution of the real signal u($\boldsymbol{x}$). Using Eq.(21) we get

$$\psi_*(\boldsymbol{x}) = \psi_1(\boldsymbol{x})**\psi_1(\boldsymbol{x}) = 4\ \psi_\delta(\boldsymbol{x})**u_*(\boldsymbol{x})\tag{24}$$

Similarly, let us denote by $\psi_{q*}(\boldsymbol{x}) = \psi_q(\boldsymbol{x})**\psi_q(\boldsymbol{x})$ the auto-convolution of the quaternionic signal (14). Using the Eq.(22) we get

$$\psi_{q*}(\boldsymbol{x}) = \psi_q(\boldsymbol{x})**\psi_q(\boldsymbol{x}) = 2\ \psi_{q\delta}(\boldsymbol{x})**u_*(\boldsymbol{x})\tag{25}$$

Similarly,

$$\psi_{M*}(\boldsymbol{x}) = \psi_M(\boldsymbol{x})**\psi_M(\boldsymbol{x}) = 2\ \psi_{M\delta}(\boldsymbol{x})**u_*(\boldsymbol{x})\tag{25}$$

7. Conclusions

Analogously to the analytic complex delta distribution defined in [7] analogous distributions have beee defined for the quaternionic [8] and monogenic signals. All these signals can be written in the form of a convolution of a real signal with the appropriate complex delta distribution. Let us mention, that the Eq.(4) applies for any dimensions , since the n-D analytic signals are a natural extension of the Gabors 1-D analytic signal from the point of view of the n-D analytic functions given by the n-D Cauchy integral. [9].

References

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Appendix 1

Consider a 2-D real signal $u(x_1, x_2)$. Its total Hilbert transform is defined by the integral

$$v(x_1, x_2) = \mathrm{H}[u(x_1, x_2)] = \frac{1}{\pi^2} \mathrm{P} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{u(\eta_1, \eta_2)\, \mathrm{d}\eta_1 \mathrm{d}\eta_2}{(\eta_1 - x_1)(\eta_2 - x_2)} \qquad \text{(A1)}$$

where P denotes the Cauchy principal value of the inegral. The partial Hilbert transform in respect to the variable x_1 is defined by the integral

$$v_1(x_1, x_2) = \mathrm{H}_1[u(x_1, x_2)] = -\frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{u(\eta_1, x_2)}{\eta_1 - x_1} \mathrm{d}\eta_1 \qquad \text{(A2)}$$

and in respect to the variable x_2 by the integral

$$v_2(x_1,x_2) = \mathrm{H}_2[u(x_1,x_2)] = -\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{u(x_1,\eta_2)}{\eta_2 - x_2}\mathrm{d}\eta_2 \quad \text{(A3)}$$

.**Appndix 2**

By the derivations of the autoconvolutions of the complex delta distributions we apply the following formulae

$\delta(x_1, x_2) = \delta(x_1, x_2)^{**}\,\delta(x_1, x_2)$, $\frac{1}{\pi x_1} * \frac{1}{\pi x_1} = -\delta(x_1)$, $\frac{1}{\pi^2 x_1 x_2} * \frac{1}{\pi x_1} = -\frac{\delta(x_1)}{\pi x_2}$

Notice, that convolutions are commutative, that is, $u(\boldsymbol{x})^{**}v(\boldsymbol{x}) = v(\boldsymbol{x})^{**}u(\boldsymbol{x})$ and associative, that is $[u(\boldsymbol{x})^{**}v(\boldsymbol{x})]^{**}w(\boldsymbol{x}) = u(\boldsymbol{x})^{**}[v(\boldsymbol{x})]^{**}w(\boldsymbol{x})]$.

Appendix 3

The monogenic complex delta distribution given by the Eq.(19)

$$\psi_{\mathrm{M}\delta}(x_1,x_2) = \delta(x_1,x_2) - \mathrm{i}\frac{x_1}{2\pi\left(\sqrt{x_1^2+x_2^2}\right)^3} - \mathrm{j}\frac{x_2}{2\pi\left(\sqrt{x_1^2+x_2^2}\right)} == \delta - \mathrm{i}\theta_{r1} - \mathrm{j}\theta_{r2}$$

and its auto-convolution is

$\psi_{\mathrm{M}\delta}(\boldsymbol{x})^{**}\psi_{\mathrm{M}\delta}(\boldsymbol{x})$

$= [\delta - \mathrm{i}\theta_{r1} - \mathrm{j}\theta_{r2}]^{**}[\delta - \mathrm{i}\theta_{r1} - \mathrm{j}\theta_{r2}] = \delta^{**}\delta - \theta_{r1}{}^{**}\theta_{r1} - \theta_{r2}{}^{**}\theta_{r2}$

$-\mathrm{i}\delta^{**}\theta_{r1} - \mathrm{j}\delta^{**}\theta_{r2} - \mathrm{i}\theta_{r1}{}^{**}\delta + \mathrm{ij}[\theta_{r1}{}^{**}\theta_{r2}\} - \mathrm{j}\delta^{**}\theta_{r2} + \mathrm{ji}\theta_{r2}{}^{**}\theta_{r1}$

The terms with ij and ji cancel. Let us investigate the term $-\theta_{r1}{}^{**}\theta_{r1} - \theta_{r2}{}^{**}\theta_{r2}$. Due to the Fourier relations (17), the Fourier transform of this term is

$$\mathrm{F}\{-\theta_{r1}{}^{**}\theta_{r1} - \theta_{r2}{}^{**}\theta_{r2}\} = \frac{f_1^2}{f_1^2+f_2^2} + \frac{f_2^2}{f_1^2+f_2^2} = 1$$

and the inverse Fourier transform yields $\delta(x_1, x_2)$..