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# Comparison of amplitude and phase functions of two-dimensional analytic and quaternionic signals

by

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## Introduction

Consider a two-dimensional real signal $u(x_1, x_2)$ defined in the Cartesian signal plane $\boldsymbol{x}=(x_1, x_2)$ and its Fourier spectrum defined by the integral

$$U(f_1, f_2) = = \mathrm{F}[u(x_1, x_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1, x_2)\exp[-\mathrm{j}2\pi(x_1 f_1 + x_2 f_2)]\,\mathrm{d}x_1\mathrm{d}x_2 \qquad (1)$$

The quaternionic spectrum is defined by the integral

$$U_q(f_1, f_2) = \mathrm{F}_q[u(x_1, x_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1, x_2)\exp(-\mathrm{i}2\pi f_1 x_1)\exp(-\mathrm{j}2\pi f_2 x_2)\,\mathrm{d}x_1\mathrm{d}x_2 \qquad (2)$$

It applies two imaginary units i and j, each equal sqr(-1) in comparison to a single unit j in (1). Let us remind that the Fourier Cartesian frequency plane $\boldsymbol{f} = (f_1, f_2)$ consists of four quadrants depicted in Fig.1. The multiplication of the spectrum $U(f_1, f_2)$ with the operator

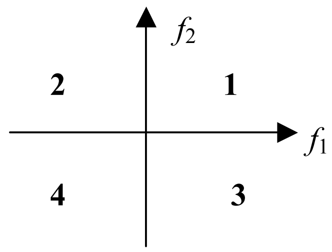



Fig.1. The four quadrants of the plane $(f_1, f_2)$

$[1 + \mathrm{sgn}(f_1)]\ [1 + \mathrm{sgn}(f_2)]$ yields the single-quadrant spectrum of the analytic signal with the support in the first quadrant. The inverse Fourier transform of this spectrum yields the following form of the analytic signal [1], [2]

$$\psi_1(x_1, x_2) = u(x_1, x_2) - v(x_1, x_2) + \mathrm{j}[v_1(x_1, x_2) + v_2(x_1, x_2)] = A_1(x_1, x_2)\exp[\Phi_1(x_1, x_2)] \qquad (3)$$

Similarly, the multiplication with the operator $[1 + \mathrm{sgn}(f_1)]\ [1 - \mathrm{sgn}(f_2)]$ yields a single-quadrant spectrum with the support in the third quadrant. The inverse transform yields the analytic signal

$$\psi_3(x_1, x_2) = u(x_1, x_2) + v(x_1, x_2) + \mathrm{j}[v_1(x_1, x_2) - v_2(x_1, x_2)] = A_2(x_1, x_2)\exp[\Phi_2(x_1, x_2)] \qquad (4)$$

which differs from $\psi_1$ only by signs. Here the function $v(x_1, x_2)$ is the total Hilbert transform of $u(x_1, x_2)$. The functions $v_1(x_1, x_2)$ and $v_2(x_1, x_2)$ are partial Hilbert transforms with respect to $x_1$ or $x_2$ (see Appendix). Due to the Hermitian symmetry of the Fourier spectrum $U(f_1, f_2)$ the real signal can be reconstructed from the half-plane spectrum, for example the half-plane $f_1 > 0$. The reconstruction formula is

$$u(x_1, x_2) = 0.5[A_1\cos(\Phi_1) + A_2\cos(\Phi_2)] \qquad (5)$$

We observe, that the real signal $u(x_1, x_2)$ should be reconstructed from the sum of two analytic signals $\psi_1(x_1, x_2) + \psi_3(x_1, x_2)$. It is so, since the two amplitudes and phase functions are different. They are given by the formulae

$$\mathrm{A}_1 = \sqrt{u^2 + v^2 + v_1^2 + v_2^2 + 2(v_1 v_2 - uv)} \;\; ; \;\; \mathrm{A}_2 = \sqrt{u^2 + v^2 + v_1^2 + v_2^2 - 2(v_1 v_2 - uv)} \qquad (6)$$

$$\tan(\Phi_1) = \frac{v_1 + v_2}{u - v} \qquad ; \qquad \tan(\Phi_2) = \frac{v_1 - v_2}{u + v}, \qquad (7)$$

that is,

$$\Phi_1 = \mathrm{Arctg}\left(\frac{v_1 + v_2}{u - v}\right) \qquad ; \qquad \Phi_1 = \mathrm{Arctg}\left(\frac{v_1 + v_2}{u - v}\right) \qquad (8)$$

The notation by capital A indicates the multi-branch character of the arctg function. We observe, that the amplitudes $\mathrm{A}_1$ and $\mathrm{A}_2$ differ by the factor $2(v_1 v_2 - uv)$. The energies of the analytic signals $\psi_1(x_1, x_2)$ and $\psi_3(x_1, x_2)$ are given by the formulae

$$\mathrm{E}_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{A}_1^2 \, dx_1 dx_2 \quad ; \quad \mathrm{E}_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{A}_2^2 \, dx_1 dx_2 \qquad (9)$$

Evidently, the energy difference is given by the formula [4]

$$\Delta\mathrm{E} = \mathrm{E}_1 - \mathrm{E}_3 = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_1 v_2 - uv) \, dx_1 dx_2 \qquad (10)$$

Let us introduce the notion of NED- the Normalized Energy Difference given by the equation

$$\mathrm{NED} = \Delta\mathrm{E}/(\mathrm{E}_1 + \mathrm{E}_2) \qquad (11)$$

***Separable 2-D signals***

A separable 2-D signal has the form of a product of 1-D signals, that is, $u(x_1, x_2) = f_1(x_1) f_2(x_2)$. For separable signals the factor $2(v_1 v_2 - uv)$ equals zero. Therefore, both amplitudes are equal and given by

$$\mathrm{A} = \mathrm{A}_1 = \mathrm{A}_2 = \sqrt{u^2 + v^2 + v_1^2 + v_2^2} \qquad (12)$$

and the corresponding analytic signals are also separable:

$$\psi_1(x_1, x_2) = \mathrm{A}\exp\{j[\varphi_1(x_1) + \varphi_2(x_2)]\} \qquad (13)$$

$$\psi_3(x_1, x_2) = \mathrm{A}\exp\{j[\varphi_1(x_1) - \varphi_2(x_2)]\}, \qquad (14)$$

that is, the phase functions (8) have the form

$$\Phi_1(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2) \qquad ; \qquad \Phi_2(x_1, x_2) = \varphi_1(x_1) - \varphi_2(x_2) \qquad (15)$$

where $\varphi_1(x_1) = \mathrm{Tan}^{-1}(g_1/f_1)$ and $\varphi_2(x_2) = \mathrm{Tan}^{-1}(g_2/f_2)$, $g_1$, $g_2$ are the 1-D Hilbert transforms of $f_1, f_2$ . Of course, due to the Parseval's theorem for separable signals the spectral energy in all quadrants of the Fourier plane is the same.

**The quaternionic complex signal**

The quaternionic complex signal is defined [3] by the inverse quaternionic Fourier transform of the quaternionic spectrum in the first quadrant and given by the formula

$$\psi_q(x_1, x_2) = u + \mathrm{i}v_1 + \mathrm{j}v_2 + \mathrm{k}v \quad ; \quad \mathrm{ij=k} \tag{16}$$

and in the polar form by

$$\psi_q(x_1, x_2) = \mathrm{A_q} \exp(\mathrm{i}\phi_\mathrm{i}) \exp(\mathrm{j}\phi_\mathrm{j}) \exp(\mathrm{k}\phi_\mathrm{k}), \tag{17}$$

that is, by a single amplitude given by (12) and three phase functions given by the equations

$$\tan(2\phi_\mathrm{i}) = \frac{2(uv_1 + vv_2)}{u^2 - v_1^2 + v_2^2 - v^2} \quad ; \quad \tan(2\phi_\mathrm{j}) = \frac{2(uv_2 + vv_1)}{u^2 + v_1^2 - v_2^2 - v^2} \tag{18}$$

$$\sin(2\phi_\mathrm{k}) = \frac{2(uv - v_1 v_2)}{\mathrm{A_q^2}} \tag{19}$$

Therefore,

$$\phi_i = 0.5\mathrm{Atan}\left[\frac{2(\mathrm{u}v_1 + vv_2)}{u^2 - v_1^2 + v_2^2 - v^2}\right] \quad ; \quad \phi_\mathrm{j} = 0.5\mathrm{Atan}\left[\frac{2(\mathrm{u}v_2 + vv_1)}{u^2 + v_1^2 - v_2^2 - v^2}\right] \tag{20}$$

$$\phi_\mathrm{k} = 0.5\mathrm{Arcsin}\left[\frac{2(uv - v_1 v_2)}{\mathrm{A_q^2}}\right] \tag{21}$$

***Derivation of the relations between the phase angles of the analytic signals and quaternionic complex signals.***

The insertion of the Eq.(19) in (6) yields the relations between the amplitudes:

$$\mathrm{A}_1 = \mathrm{A_q}\sqrt{1 + \sin(2\phi_\mathrm{k})} \quad ; \quad \mathrm{A}_2 = \mathrm{A_q}\sqrt{1 - \sin(2\phi_\mathrm{k})} \tag{22}$$

Therefore,

$$\mathrm{A_q} = \sqrt{\frac{\mathrm{A}_1^2 + \mathrm{A}_2^2}{2}} \tag{23}$$

$$\sin(2\phi_\mathrm{k}) = \frac{\mathrm{A}_1^2 - \mathrm{A}_2^2}{\mathrm{A}_1^2 + \mathrm{A}_2^2} \quad ; \quad \phi_\mathrm{k} = 0.5\arcsin\left(\frac{\mathrm{A}_1^2 - \mathrm{A}_2^2}{\mathrm{A}_1^2 + \mathrm{A}_2^2}\right) \tag{24}$$

The above formula shows, that the angle $\phi_\mathrm{k} \in (-\pi/4, \pi/4)$ and the values $\pm\pi/4$ occur only, if $\mathrm{A}_1$ or $\mathrm{A}_2$ equal zero. For natural images the probability that one of the amplitudes vanishes is very small. The addition of the tangent functions given by (18) yields

$$2(uv_1 + vv_2) = (u^2 - v^2)[\tan(\Phi_1) + \tan(\Phi_2)] \tag{25}$$

and the subtraction

$$2(uv_2 + vv_1) = (u^2 - v^2)[\tan(\Phi_1) - \tan(\Phi_2)] \tag{26}$$

Therefore, the phase functions of the quaternionic complex signal are related to the phase functions $\Phi_1$ and $\Phi_2$ by the equations

$$\tan(2\phi_i) = [\tan(\Phi_1) + \tan(\Phi_2)]\frac{1}{1+(v_2^2 - v_1^2)/(u^2 - v^2)} = \tan(\Phi_1 + \Phi_2) \qquad (27)$$

$$\tan(2\phi_j) = [\tan(\Phi_1) - \tan(\Phi_2)]\frac{1}{1-(v_2^2 - v_1^2)/(u^2 - v^2)} = \tan(\Phi_1 - \Phi_2) \qquad (28)$$

and we get the simple relations

$$\phi_i = 0.5(\Phi_1 + \Phi_2) \quad ; \quad \phi_j = 0.5(\Phi_1 - \Phi_2) \qquad (29)$$

The energy difference defined by the Eq.(10) may be written using the Eq.(19) in the form

$$\Delta E = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_q^2 \sin(2\phi_k)\, dx_1 dx_2 \qquad (30)$$

For separable signals or for signals with circular symmetry $\phi_k$ equals zero and $\Delta E = 0$.

Let us mention, that separability is relative in respect to the rotation of the coordinate system. A signal separable in a non-rotated coordinate system may be non-separable in the rotated system. For separable signals the phase functions given by Eq.(29) take the form

$$\phi_i = \varphi_1 \; ; \quad \phi_j = \varphi \; ; \quad \phi_k = 0 \qquad (31)$$

For quaternionic signals the reconstruction formula ( see Eq.5) nas the form

$$u(x_1, x_2) = A_q[\cos(\phi_i)\cos(\phi_j)\cos(\phi_k) - \sin(\phi_i)\sin(\phi_j)\sin(\phi_k) \qquad (31)$$

**Example 1**

Consider a real signal in the form of a product of two cosine functions

$$u(x_1, x_2) = \cos(2\pi f_1 x_1)\cos(2\pi f_2 x_2) \qquad (32)$$

The corresponding analytic signal defined by the Eq.(3) is (33)

$$\psi_1(x_1, x_2) = \exp[j2\pi(f_1x_1 + f_2x_2)] = \exp(j\phi_1)\exp(j\phi_2) \qquad (34)$$

Evidently, $\varphi_1 = 2\pi f_1x_1 + 2\pi f_2x_2$ and $\varphi_2 = 2\pi f_1x_1 - 2\pi f_2x_2$. We have $A = A_q = 1$, $\phi_i = 2\pi f_1x_1$, $\phi_j = 2\pi f_2x_2$ and of course $\phi_k = 0$. The phase functions are displayed in Fig.1.

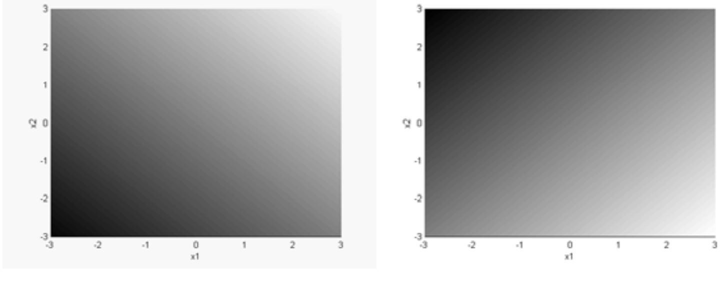



Fig.1a. The phase functions of the analytic signal: Left $\Phi_1$ , right $\Phi_2$.

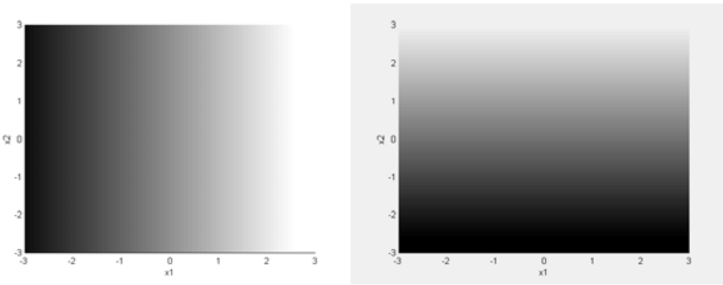


Fig.1b.The quaternionic phase functions: Left $\phi_i$, right $\phi_j$.

**Example 2**

Consider the 2-D complex delta distribution [5]

$$\psi_\delta\,(x_1, x_2) = [\delta(x_1) + j(1/(\pi x_1)]\otimes[\delta(x_2) + j(1/(\pi x_2)] \qquad (35)$$

where ⊗ denotes the tensor product of distributions. Let us omit this symbol in next notations. The local amplitude has the form

$$A_\delta\,(x_1, x_2) = 1/(\pi^2|x_1||x_2|) \qquad (36)$$

And the phase functions are

$$\Phi(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2) = 0.5\pi\mathrm{sgn}(x_1) + 0.5\pi\mathrm{sgn}(x_2) \qquad (37)$$

Let us remind, that the 2-D analytic signal (3) may be written in the convolution form

$$\psi_1(x_1, x_2) = u(x_1, x_2)^{**}\ \psi_\delta\,(x_1, x_2) \qquad (38)$$

Let us define the quaternionic complex delta distribution

$$\psi_{\delta q}\,(x_1, x_2) = \delta\,(x_1, x_2) + i\delta(x_2)/(\pi x_1) + j\delta(x_1)/(\pi x_2) + k[1/(\pi^2 x_1 x_2)] \qquad (39)$$

The local amplitude is given by the Eq.(36), and the phase functions are

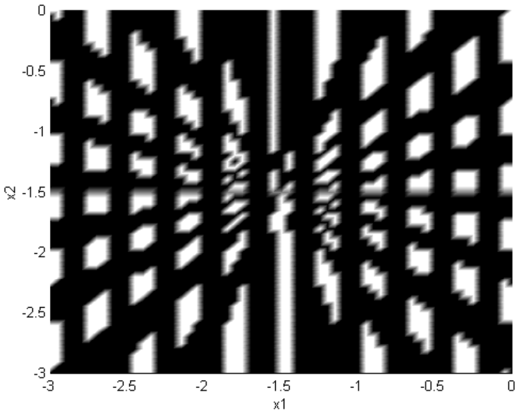
$$\phi_i(x_1, x_2) = \varphi_1(x_1) = 0.5\pi\mathrm{sgn}(x_1)\ ;\ \phi_j(x_1, x_2) = \varphi_2(x_2) = 0.5\pi\mathrm{sgn}(x_2) \qquad (40)$$

Analogously to (38), the quaternionic complex signal (16) may be written in the convolution form

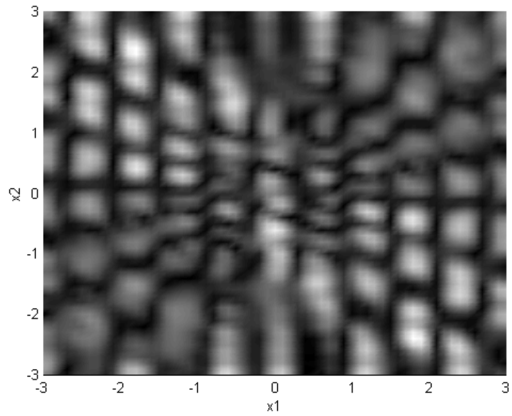
$$\psi_q(x_1, x_2) = u(x_1, x_2)^{**}\ \psi_{\delta q}\,(x_1, x_2) \qquad (41)$$

**Example 3**

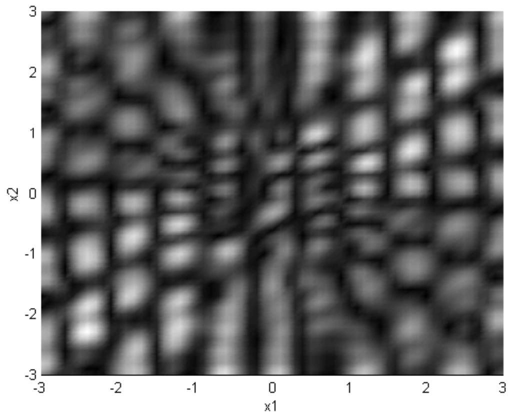
This example is illustrated only by images.



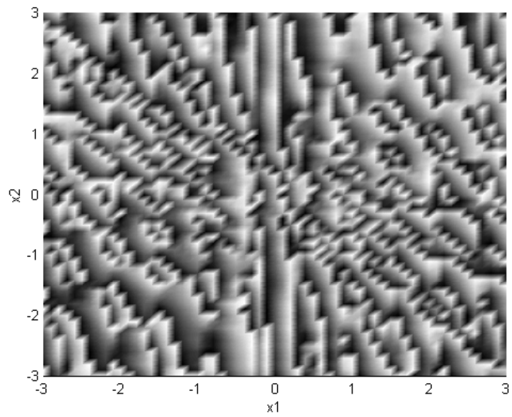

Original image



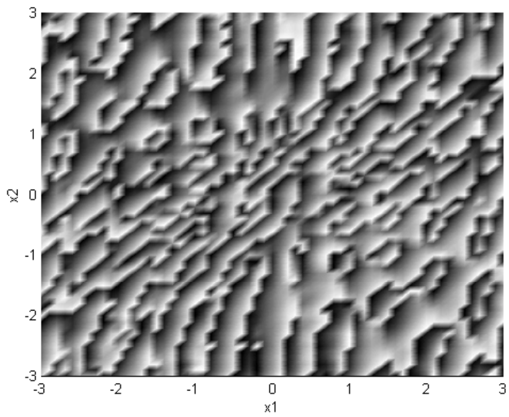

The amplitude $A_1$ of the analytic signal $\psi_1$



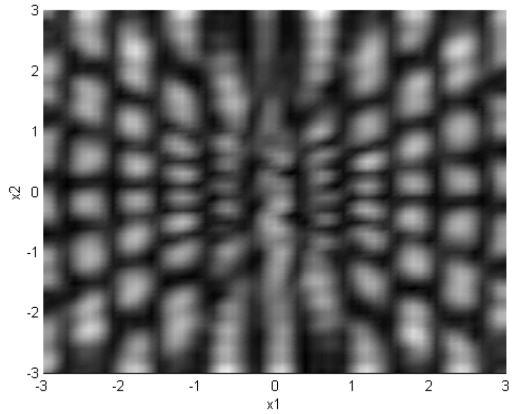

The amplitude $A_2$ of the analytic signal $\psi_3$



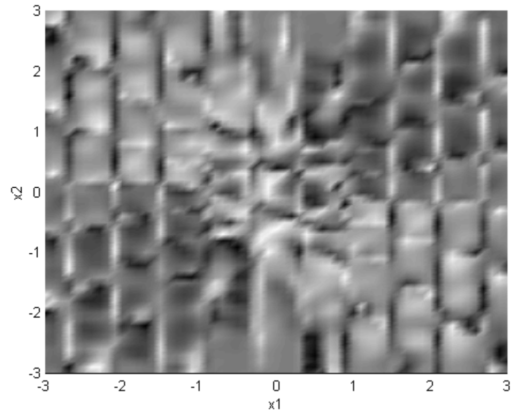

The phase function $\Phi_1$ of the analytic signal $\psi_1$



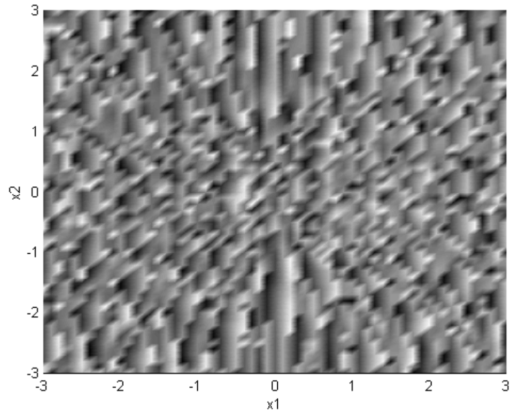

The phase function $\Phi_2$ of the analytic signal $\psi_3$



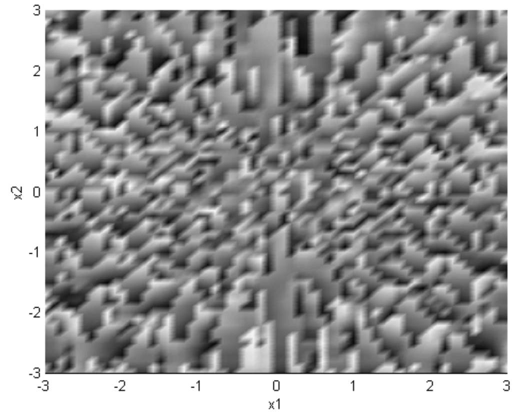

The quaternionic amplitude $A_q$



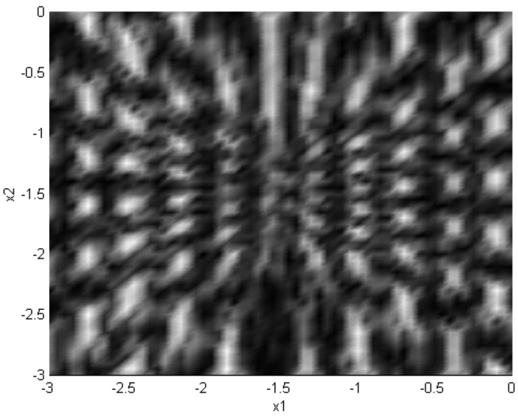

The quaternionic phase $\phi_k$



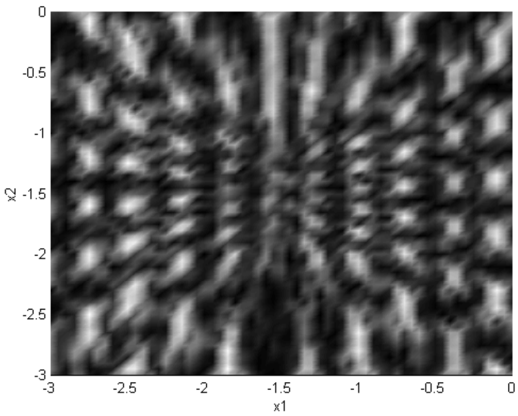

The quaternionic phase $\phi_i = 0.5(\Phi_1 + \Phi_2)$




The quaternionic phase $\phi_j = 0.5(\Phi_1 - \Phi_2)$




Reconstruction from analytic signals, Eq.(6)




Quaternionic reconstruction, Eq.(31)

The original signal is non-separable. In consequence the quaternionic phase $\phi_k$ (see Eq.24) differs from zero. However, the quaternionic amplitude $A_q$ (see Eq.(23)) does not differ much from the two amplitudes $A_1$ and $A_2$ of the analytic signals $\psi_1$ and $\psi_3$. Remark: The phase jumps of the presented phase functions are not removed

## Conclusion

A real signal may be represented by two analytic signals, that is by two amplitudes and two phase functions or alternatively by a single quaternionic complex signal, that is, by a single amplitude and three phase functions. We derived very simple formulae relating the quaternionic and analytic representations of the above amplitude and phase functions.

## References


[1] Hahn S.L., Multidimensional complex signals with single-orthant spectra, Proc.IEEE, vol.80, No.8, August 1992, pp.1287-1300.

[2] Hahn S.L. Hilbert Transforms in Signal Processing, Artech House, 1996.

[3] Bülow T., Hypercomplex spectral signal representation for the processing and analysis of images, Institut für Informatik und Praktische Mathematik, Christian-Albrechts-Universität Kiel, Bericht Nr.9903, August 1999.

[4] Hahn S.L., Buchowicz A., Wigner-Ville distributions of complex multidimensional analytic signals, A paper delivered on 21 May 1997 to the Editor of the Proc.IEEE, not accepted for publication In this paper the authors introduced the notion of NED.

[5] Hahn S.L., The *N*-Dimensional Complex Delta Distribution, IEEE Trans.on Signal Processing, vol.44, NO.7, July 1996, pp.1833-1837.


## Appendix

Consider a 2-D real signal $u(x_1, x_2)$. Its total Hilbert transform is defined by the integral

$$v(x_1,x_2)=\mathrm{H}[u(x_1,x_2)]=\frac{1}{\pi^2}\mathrm{P}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{u(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2}{(\eta_1-x_1)(\eta_2-x_2)} \qquad \text{(A1)}$$

where P denotes the Cauchy principal value of the inegral. The partial Hilbert transform in respect to the variable $x_1$ is defined by the integral

$$v_1(x_1,x_2)=\mathrm{H}_1[u(x_1,x_2)]=-\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{u(\eta_1,x_2)}{\eta_1-x_1}\mathrm{d}\eta_1 \qquad \text{(A2)}$$

and in respect to the variable $x_2$ by the integral

$$v_2(x_1,x_2)=\mathrm{H}_2[u(x_1,x_2)]=-\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{u(x_1,\eta_2)}{\eta_2-x_2}\mathrm{d}\eta_2 \qquad \text{(A3)}$$

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