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AND INFORMATION TECHNOLOGY
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On the Frequency-domain Definition of the Monogenic Signal

by

Stefan L. Hahn and Kajetana M. Snopek

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Stefan L. Hahn, *Life Senior Member, IEEE,* and Kajetana M. Snopek

Institute of Radioelectronics, Warsaw University of Technology

Nowowiejska 15/19, 00-665 Warsaw, Poland

Tel: (+48) (22) 6607647, Fax: (+48) (22) 8255248

E-mail: hahn@ire.pw.edu.pl, snopek@ire.pw.edu.pl

Abstract - This paper presents the frequency-domain definition of the monogenic signal. It is derived basing on the notion of the Quaternionic Fourier Transform.

Keywords: Quaternionic Fourier Transform, quaternionic and monogenic signals

I. INTRODUCTION

In a very interesting recent paper [1] the authors defined a new notion of the 2-D monogenic signal. It is derived using Riesz transforms called also "isotropic Hilbert transforms" [2]. Its signal-domain definition has the form

$$\psi_M(x_1,x_2)=u(x_1,x_2)+iv_{r1}(x_1,x_2)+jv_{r2}(x_1,x_2) \tag{1}$$

where u is a 2-D real signal, v_{r1} and v_{r2} are the corresponding Riesz transforms. They have the form of convolutions of u with so called Riesz kernels r_1 and r_2, i.e.,

$$v_{r1}(x_1,x_2)=u(x_1,x_2)**r_1(x_1,x_2), \quad v_{r2}(x_1,x_2)=u(x_1,x_2)**r_2(x_1,x_2). \tag{2}$$

Riesz kernels and their 2-D Fourier transforms are given by the relations

$$r_1(x_1,x_2)=\frac{x_1}{2\pi\left[\sqrt{x_1^2+x_2^2}\right]^3}\overset{2-D\ FT}{\Leftrightarrow}\frac{-jf_1}{\sqrt{f_1^2+f_2^2}}, \quad r_2(x_1,x_2)=\frac{x_2}{2\pi\left[\sqrt{x_1^2+x_2^2}\right]^3}\overset{2-D\ FT}{\Leftrightarrow}\frac{-jf_2}{\sqrt{f_1^2+f_2^2}}. \tag{3}$$

In the chapter II we derive the frequency-domain definition of the monogenic signal using the inverse Quaternionic Fourier Transform (*QFT*) of a specific spectrum. The Appendix contains the definition and basic properties of the *QFT*.

II. THE FREQUENCY-DOMAIN DEFINITION OF THE MONOGENIC SIGNAL

A real signal u can be written in the form of a union of four terms: $u(x_1, x_2) = u_{\text{ee}} + u_{\text{oe}} + u_{\text{eo}} + u_{\text{oo}}$, where the subscripts "*e*" and "*o*" denote evenness or oddness of a function w.r.t. the corresponding variable. With this decomposition the 2-D Fourier transform has the form

$$U(f_1, f_2) = \iint (u_{ee} + u_{oe} + u_{eo} + u_{oo})\ e^{-j2\pi(f_1x_1 + f_2x_2)} dx_1 dx_2 = U_{ee} - U_{oo} - j(U_{oe} + U_{eo}) \tag{4}$$

where

$$U_{ee}(f_1, f_2) = \iint u_{ee}(x_1, x_2) \cos(2\pi f_1 x_1) \cos(2\pi f_2 x_2) dx_1 dx_2 = U_{ee}(-f_1, f_2), \tag{5}$$

$$U_{oo}(f_1, f_2) = \iint u_{oo}(x_1, x_2) \sin(2\pi f_1 x_1) \sin(2\pi f_2 x_2) dx_1 dx_2 = -U_{oo}(-f_1, f_2), \tag{6}$$

$$U_{oe}(f_1, f_2) = \iint u_{oe}(x_1, x_2) \sin(2\pi f_1 x_1) \cos(2\pi f_2 x_2) dx_1 dx_2 = -U_{oe}(-f_1, f_2), \tag{7}$$

$$U_{eo}(f_1, f_2) = \iint u_{eo}(x_1, x_2) \cos(2\pi f_1 x_1) \sin(2\pi f_2 x_2) dx_1 dx_2 = U_{eo}(-f_1, f_2). \tag{8}$$

The QFT of u may be calculated using the form [3]

$$U_q(f_1, f_2) = QFT(u) = \frac{1-k}{2} U(f_1, f_2) + \frac{1+k}{2} U(-f_1, f_2) \tag{9}$$

Notice that if $U(f_1, f_2) = U(-f_1, f_2)$, then $U_q = U$. The insertion of $U(f_1, f_2)$ and $U(-f_1, f_2)$ yields

$$U_q(f_1, f_2) = U_{ee} - jU_{eo} + k(U_{oo} - jU_{oe}) = U_{ee} - iU_{oe} - jU_{eo} + kU_{oo}. \tag{10}$$

The quaternionic spectrum of the first Riesz term is

$$QFT[v_{r1}] = V_{r1}(f_1, f_2) = (-iS_1) \cdot U_q \tag{11}$$

where $S_1 = \dfrac{f_1}{\sqrt{f_1^2 + f_2^2}}$. The insertion of (10) yields

$$QFT(v_{r1}) = (-iU_{ee} - U_{oe} + jU_{oo} + kU_{eo}) S_1. \tag{12}$$

Therefore,

$$iQFT(v_{r1}) = (U_{ee} - iU_{oe} - jU_{eo} + kU_{oo}) S_1 = U_q \cdot S_1. \tag{13}$$

The quaternionic spectrum of the second Riesz term is

$$QFT[v_{r2}] = V_{r2}(f_1, f_2) = U_q \cdot (-jS_2) \tag{14}$$

where $S_2 = \dfrac{f_2}{\sqrt{f_1^2 + f_2^2}}$. Note the inverse order in comparison to (11). The insertion of (10) yields

$$QFT(v_{r2}) = (-jU_{ee} - U_{eo} - iU_{oo} - kU_{oe}) S_2. \tag{15}$$

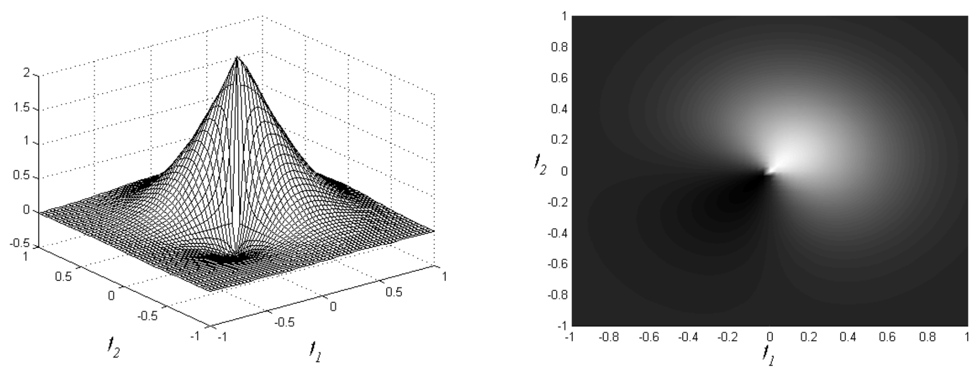
Therefore,

$$jQFT\left(v_{r2}\right)=\left(U_{ee}-iU_{eo}-jU_{eo}+kU_{oo}\right)S_2=U_q\cdot S_2\,. \tag{16}$$

The addition of (10), (13) and (16) yields the following quaternionic spectrum of the monogenic signal

$$\begin{aligned} QFT\left[\psi_M\right]=\Gamma_M\left(f_1,f_2\right)&=U_q\left(1+S_1+S_2\right) \\ &=U_q\left[1+\frac{f_1+f_2}{\sqrt{f_1^2+f_2^2}}\right]. \end{aligned} \tag{17}$$

where U_q is given by (10). Fig.1 shows an example of the spectrum (17) with $u\left(x_1,x_2\right)=u_{ee}\left(x_1,x_2\right)=\exp\left[-\pi\left(x_1^2+x_2^2\right)\right]$ corresponding to $U_q\left(f_1,f_2\right)=U_{ee}\left(f_1,f_2\right)=\exp\left[-\pi\left(f_1^2+f_2^2\right)\right]$. The total energy of the spectrum of Fig.1 equals 1 (twice the energy of the Gaussian signal which equals 0.5). The energy of the spectrum with the support in the first quadrant (f_1>0, f_2>0) equals ≈ 0.69 and in the fourth quadrant (f_1<0, f_2<0) only 0.0123.



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REFERENCES

[1] Felsberg M., Sommer G., "The Monogenic Signal," *IEEE Trans. Signal Processing*, vol.49, No. 12, pp. 3136-3140, Dec. 2001.

[2] Larkin K.G., Bone D., Oldfield M.A., "Natural demodulation of two-dimensional fringe patters: I. General background of the spiral phase quadrature transform," *J. Opt. Soc. Am.*, vol.18, No.8, pp.1862-1870, Aug. 2001.

[3] Sommer G. (Ed.), *Geometric Computing with Clifford Algebra. Theoretical Foundations and Applications in Computer Vision and Robotics*, ISBN 3-540-41198-4, Springer Verlag Berlin Heidelberg 2001. Remark: Elements of the derivation of (10) are given in Chapters 9 and 10. However, the final form of (10) is delivered to the authors in a private communication of Dr.Michael Felsberg.

[4] Bülow T., Sommer G., "The Hypercomplex Signal - A Novel Extension of the Analytic Signal to the Multidimensional Case," *IEEE Trans. Signal Processing*, vol. 49, No. 11, pp. 2844-2852, Nov. 2001.

[5] Bülow T., "Hypercomplex spectral signal representation for the processing and analysis of images," in Bericht Nr. 9903, Institut für Informatik und Praktische Mathematik, Christian-Albrechts-Universität Kiel, Aug. 1999 (http://www.cis.upenn.edu/~thomasbl/thesis.html).

APPENDIX

A. 2-D Quaternionic Signals

The concept of the quaternion number was introduced by Hamilton in 1843 and defined by the formula

$$q = 1a + ib + jc + kd \tag{A.1}$$

where $a, b, c, d \in \Re$ and the products of the imaginary units i, j, k obey the rules presented in Table 1. These products are non-commutative, i.e., $ij \neq ji$.

TABLE 1.

THE PRODUCTS OF IMAGINARY UNITS

1	i	j	k
i	$i^2 = -1$	$ij = k$	$ik = -j$
j	$ji = -k$	$j^2 = -1$	$jk = i$
k	$ki = j$	$kj = -i$	$k^2 = -1$

The conjugate quaternion number is given by

$$q^* = 1a - ib - jc - kd \tag{A.2}$$

where the numerical coefficient 1 is usually omitted in notations. The norm (modulus) of q is defined as

$$|q| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}\ . \tag{A.3}$$

Replacing in (A.1) the numbers a, b, c, d by real functions $a(x_1,x_2)$, $b(x_1,x_2)$, $c(x_1,x_2)$ and $d(x_1,x_2)$, yields a quaternion-valued function of the form

$$q(x_1,x_2) = a(x_1,x_2) + ib(x_1,x_2) + jc(x_1,x_2) + kd(x_1,x_2), \tag{A.4}$$

B. The Quaternionic Fourier Transform

The Quaternionic Fourier Transform (*QFT*), that is, the quaternion-valued spectrum of the real signal $u(x_1,x_2)$, is defined by the integral [3], [4]

$$U_q(f_1,f_2) = QFT\left[u(x_1,x_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi f_1 x_1} u(x_1,x_2) e^{-j2\pi f_2 x_2} dx_1 dx_2 \ . \tag{A.5}$$

The first exponential uses the imaginary unit i and the second one, the unit j. Due to the non-commutativity of products of quaternions, the change of the order of functions in the integrand of (A.5) yields another quaternion valued function, which differs by signs of the terms. The *QFT* is invertible and its inverse is given by the integral

$$u(x_1,x_2) = QFT^{-1}\left[U_q(f_1,f_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i2\pi f_1 x_1} U_q(f_1,f_2) e^{j2\pi f_2 x_2} df_1 df_2 \ . \tag{A.6}$$

C. The Quaternionic Hermitian Symmetry

The quaternionic spectrum defined by the *QFT* obeys the rules of the quaternionic Hermitian symmetry defined by the relations [3], [4], [5]

$$U_q(-f_1,f_2) = \alpha_j\left[U_q(f_1,f_2)\right], \tag{A.7}$$

$$U_q(-f_1,-f_2) = \alpha_k\left[U_q(f_1,f_2)\right], \tag{A.8}$$

$$U_q(f_1,-f_2) = \alpha_i\left[U_q(f_1,f_2)\right] \tag{A.9}$$

where the functions α_i, α_j and α_k are called *involutions* of U_q. Using (10) the involutions α_i , α_j and α_k are defined as follows:

$$\alpha_i(f_1,f_2) = -iU_q i = U_{ee} - iU_{oe} + jU_{eo} - kU_{oo}, \tag{A.10}$$

$$\alpha_j(f_1,f_2) = -jU_q j = U_{ee} + iU_{oe} - jU_{eo} - kU_{oo}, \tag{A.11}$$

$$\alpha_k(f_1,f_2) = -kU_q k = U_{ee} + iU_{oe} + jU_{eo} + kU_{oo}. \tag{A.12}$$

These involutions can be easily derived using the rules of Table 1. The quaternionic Hermitian symmetry is illustrated in Fig.2. Any real signal can be reconstructed using a single-quadrant quaternionic spectrum. The spectral information in the complementary three quadrants is redundant.

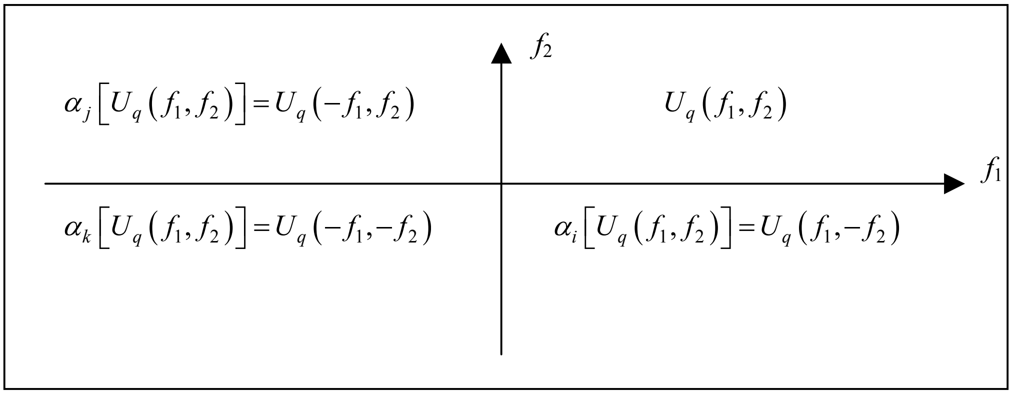

Fig. 2. The quaternionic Hermitian symmetry