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of multidimensional analytic functions**

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Abstract

Starting with the Cauchy integral representation of a multidimensional analytic functions it was shown that complex signals with single-orthant spectra are boundary distributions of multidimensional analytic functions.

Multi-dimensional analytic functions

Let C be a complex Cartesian plane. The Cartesian product

$$C^n = C \times C \times \dots \times C \quad (\text{n-times}) \quad (1)$$

defines the n-dimensional complex space. Each point of this space is defined by the values of n complex coordinates $z = (z_1, z_2, \dots, z_n)$; $z_k = x_k + j y_k$. A complex-valued function $f(z)$ defined on the open set $D \subset C^n$ is analytic at the $z_0 \in D$ if in the neighborhood of z_0 can be expanded into the power series, that is,

$$f(z) = \sum_{\alpha_1, \dots, \alpha_n=0}^{\infty} f_{\alpha_1, \dots, \alpha_n}(z_0) (z_1 - z_{01})^{\alpha_1} (z_2 - z_{02})^{\alpha_2} \dots (z_n - z_{0n})^{\alpha_n} \quad (2)$$

The n-dimensional Cauchy integral

An n-dimensional function analytic in the region $D = D_1 \times D_2 \times \dots \times D_n$ is represented by the n-dimensional Cauchy integral

$$f(z) = \frac{1}{(2\pi j)^n} \oint_{\partial D_1} d\zeta_1 \dots \oint_{\partial D_n} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_n}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} \quad ; \quad z \in D \quad (3)$$

where $\partial D_1, \dots$, are closed contours of integration in D_1, \dots, D_n . Starting with this integral representation let us show, that n-dimensional complex signals with single orthant spectra [Hahn, 1992] represent a boundary distribution of analytic functions. We start with the one-dimensional case.

The one-dimensional analytic signal as boundary distribution of a one-dimensional analytic function.

The Cauchy integral representation of a one-dimensional analytic function $\psi(z)$, $z = t + j \tau$, (we change the notation $z = x + jy$ to the more common notation for time signals) is

$$\psi(z) = \frac{1}{2\pi j} \oint_C \frac{\psi(\zeta)}{\zeta - z} d\zeta \quad (4)$$

Consider a function analytic inside the region D located in the upper half complex plane and bounded by a closed curve C , as shown in Fig.1. The curve C is divided in three parts: A large half-circle C_R of radius R , a line C_t parallel to the real axis shifted by ε and a small half-circle C_ε of radius ε centered the point $z_0 = t_0 + j\varepsilon$. Using anti-clock direction of integration, the Cauchy integral (4) takes the form

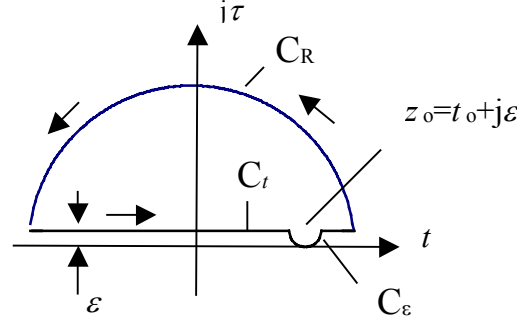


Fig.1 The contour of integration along the closed contour $C = C_R + C_t + C_\varepsilon$ of the complex plane $(t, j\tau)$.

$$\psi(z_0) = \frac{1}{2\pi j} \left\{ \int_R^{0-\varepsilon} (\bullet) + \int_{0+\varepsilon}^R \frac{\psi(\zeta)}{\zeta - z_0} d\zeta + \int_{C_R} \frac{\psi(\zeta)}{\zeta - z_0} d\zeta + \int_{C_\varepsilon} \frac{\psi(\zeta)}{\zeta - z_0} d\zeta \right\} ; z_0 = t_0 + j\varepsilon ; \quad (5)$$

The analytic signal $\psi(t)$ (Gabor, 1946) is defined using the limit of this integral with $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$ that is $z_0 \rightarrow t_0 + j0_+$. The notation 0_+ indicates, that the imaginary variable $j\tau = j\varepsilon$ reaches zero from the upper side. For analytic functions the integral along C_R vanishes for $R \rightarrow \infty$. The limit of the integral along the small half-circle equals $0.5\psi(t_0, 0_+)$, since within the infinitesimally small circle around t_0 the continuous analytic function has a constant value $\psi(t_0, 0_+)$ and the integral $\int_{C_R} \frac{dz}{z - z_0} = j\pi$. Therefore, changing the

notation $t_0 \rightarrow t$ and $\zeta \rightarrow \eta$ we get the equation

$$\frac{1}{2}\psi(t + j0_+) = u(t, 0_+) + jv(t, 0_+) = \frac{1}{2\pi j} P \int_{-\infty}^{\infty} \frac{\psi(\eta)}{\eta - t} d\eta \quad (6)$$

where P denotes the Cauchy Principal Value of the integral (see. Eq.(5)). Equating the real and imaginary parts of this equation yields the well known relations between u and v in the form of a pair of Hilbert transforms

$$v(t) = H[u(x)] = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(\eta)}{\eta - x} d\eta \quad (7)$$

$$u(t) = H^{-1}[v(x)] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(\eta)}{\eta - x} d\eta \quad (8)$$

We have shown, that the analytic signal is given by the boundary distribution of the analytic function along the 0_+ side of the real axis of the complex plane. The Fourier spectrum of this signal is one-sided at the positive frequencies. However, if we integrate along a closed path in the lower half-plane ($t, j\tau$), we get the conjugate analytic signal with the Fourier spectrum one-sided at negative frequencies. It represents the boundary distribution of the analytic function along the 0_- side of the real axis. Let us mention, that the choice of the distribution along the real axis is a matter of convention. We could derive an analytic signal given by the boundary distribution along the imaginary axis.

Complex signals with single-quadrant spectra as boundary distributions of two-dimensional analytic functions

Let us extend the above presented evidence, that 1-D signals are boundary distributions of analytic functions, for 2-D signals with single-quadrant spectra. A complex signal with a single-quadrant spectrum in the first quadrant is given by the formula (Hahn,1992)

$$\psi(x_1, x_2) = u(x_1, x_2) - v(x_1, x_2) + j[v_1(x_1, x_2) + v_2(x_1, x_2)] \quad (9)$$

where $u(x_1, x_2)$ is a real function, $v(x_1, x_2)$ is the total Hilbert transform of $u(x_1, x_2)$ of the form

$$v(x_1, x_2) = H[u(x_1, x_2)] = \frac{1}{\pi^2} \text{P} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{u(\eta_1, \eta_2) d\eta_1 d\eta_2}{(\eta_1 - x_1)(\eta_2 - x_2)} \quad (10)$$

and $v_1(x_1, x_2)$ or $v_2(x_1, x_2)$ are partial Hilbert transforms of $u(x_1, x_2)$ in respect to x_1 or x_2 , that is,

$$v_1(x_1, x_2) = H_1[u(x_1, x_2)] = -\frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{u(\eta_1, x_2) d\eta_1}{\eta_1 - x_1} \quad (11)$$

$$v_2(x_1, x_2) = H_2[u(x_1, x_2)] = -\frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{u(x_1, \eta_2) d\eta_2}{\eta_2 - x_2} \quad (12)$$

Similarly to the 1-D case we start the derivation with the 2-D Cauchy integral

$$\psi(z_1, z_2) = \frac{1}{(2\pi j)^2} \oint_{\partial D_1} \oint_{\partial D_2} \frac{\psi(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \quad (13)$$

where ∂D_1 and ∂D_2 are closed contours of integration in the complex planes D_1 and D_2 with integration with respect to $\zeta_1 = \eta_1 + j\eta_1$ or $\zeta_2 = \eta_2 + j\eta_2$ respectively. Both contours have the same shape, as displayed in Fig.1. The integration along the contour ∂D_1 yields in the limit $R_1 \rightarrow \infty$, $\varepsilon_1 \rightarrow 0_+$ (see the 1-D case) the following integral

$$\psi(x_1, z_2) = \frac{1}{2\pi j} \oint_{\partial D_2} \left\{ \frac{1}{2\pi j} \text{P} \int_{-\infty}^{\infty} \frac{\psi(\eta_1, \zeta_2) d\eta_1}{\eta_1 - x_1} + \frac{1}{2} \psi(x_1, \zeta_2) \right\} \frac{d\zeta_2}{\zeta_2 - z_2} \quad (14)$$

which can be written in the form (see Eq.(11))

$$\psi(x_1, z_2) = \frac{1}{2\pi j} \oint_{\partial D_2} \left\{ \frac{-1}{2j} H_1[\psi(x_1, \zeta_2)] + \frac{1}{2} \psi(x_1, \zeta_2) \right\} \frac{d\zeta_2}{\zeta_2 - z_2} \quad (15)$$

Next integration along the contour ∂D_2 yields

$$\psi(x_1, x_2) = \frac{1}{4\pi} P \int_{-\infty}^{\infty} \frac{H_1[\psi(x_1, \eta_2)]}{\eta_2 - x_2} d\eta_2 + \frac{j}{4} H_1[\psi(x_1, x_2)] + \frac{1}{4\pi j} P \int_{-\infty}^{\infty} \frac{\psi(x_1, \eta_2)}{\eta_2 - x_2} d\eta_2 + \frac{1}{4} \psi(x_1, x_2) \quad (16)$$

Introducing the operators of Hilbert transforms H , H_1 and H_2 (see Eqs.(10), (11) and (12), and the relation $H=H_2[H_1]$ we get using the multiplication of both sides by $4/3$ the following relation

$$\psi(x_1, x_2) = \frac{1}{3} \{ H[\psi(x_1, x_2)] + jH_1[\psi(x_1, x_2)] + jH_2[\psi(x_1, x_2)] \} \quad (17)$$

We omitted here the notation $\psi(x_1, 0_+, x_2, 0_+)$. The insertion in this equation the complex signal given by the Eq.(9) confirms the equality of (9) and (17). Concluding, we presented the evidence, that the complex signal with the with single-quadrant spectrum in the first quadrant is a boundary distribution of a 2-D analytic function. The derivation for the other three signals with spectra in the second, third and fourth quadrants is the same using only a change of signs. The extension of the derivation for n-dimensional signals is straightforward and do not change the method of derivation. In our paper in 1992 (Hahn, 1992) we used the name “complex signals with single orthant spectra” since at that time failed the evidence of analyticity in the formal sense. The presented derivation enables the use of the name “Multidimensional analytic signals with single orthant spectra”.

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