ENSEMBLE AVERAGES of WIGNER DISTRIBUTIONS of NOISE and TELECOMMUNICATION SIGNALS with EMPHASIS on the ROLE of CROSS-TERMS

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Abstract: The paper presents a study of properties of ensemble averages of Wigner time-frequency distributions (*WDs)* of random processes defined by statistically independent samples of stationary and nonstationary Gaussian noise and of radio-frequency telecommunication signals. We used samples of *PSK* and *FSK* signals transmitting random telegraph signals. The *WDs* of single samples of these random processes have the form of random bipolar fields $W(t, f)$, while ensemble averages $E\{W(t, f)\}$ are well defined deterministic functions. The *WDs* of real signals have the form of a sum of an even term and a cross-term. In selected cases, theoretical forms of ensemble averages of these terms are derived and compared with computer simulations. In other cases, only computer simulations are applied. It was shown that ensemble averages of even terms are usually (but not always) unipolar and of cross-terms bipolar well-defined deterministic functions. For so-called proper processes the ensemble averages of cross-terms are cancelled. In computer simulations, their level decreases with increasing number of samples. In other cases, cross-terms carry an information about some properties of a random process. Derivations show that the notion of a cross-term coincides with the double value of the real part of the so-called complementary Wigner distribution. In consequence, the paper yields the answer to the question, in which cases the complementary distribution matters and yields a deeper insight into the properties and role of cross-terms.

Keywords –Wigner distributions , random analytic signals and processes. ensemble averages.

EDICS: SSP-SSAN, SSP-SPEC, SSP-SNMD, SPC-APPL.

I. INTRODUCTION

The Wigner distribution (*WD*) [1] is the most popular member of time-frequency distributions. Its theoretical background and applications are described in numerous papers and summarized in many books, for example [2], and handbooks, for example [3]. In the case of random processes, we have to distinguish *WDs* of single sample functions (realizations) $W(t. \ f)$ and its ensemble averages $E\{W(t, f)\}$. Their properties are different. The *WD* of a single sample of a random process, for example a sample of Gaussian noise, has the form of a bipolar random field . Differently, the ensemble average $E\{W(t,f)\}$ is a well defined deterministic function. Generalized expressions in this context are given in [4]. The *WD* of a real signal contains so-called cross-terms (see Eq.1 in Section 2). The authors of [5] describe the role of ensemble averages of a so-called complementary *WDs* of analytic signals. However, the notion of a cross-term coincides with the double real part of the complementary *WD.* In this paper, in Section V, for selected random processes estimates of ensemble averages of cross-terms are derived or calculated yielding a deeper insight into its properties.

Let us have a comment about computer calculations of ensemble averages. Theoretically, the operator $E\{ \ \}$ requires the summation of infinite number of samples. In computer simulations, the number of samples is finite. In this paper, to get a reasonable accuracy we used 200 to 2000 statistically independent samples. It is hard to imagine that such amounts could be available from experiments. We should apply a theoretical model of generation samples of random signals. If the construction of such a model is impossible, the ensemble averages cannot be calculated. This paper is illustrated with estimates of ensemble averages of *WDs* of noise and telecommunication signals calculated using theoretical models. Let us mention that if the correlation function of the random process can be derived in a closed form, the ensemble average of the *WD* defined by the Fourier transform of the correlation function may also have a closed form. Differently, the *WD* of a single sample, for example a sample of a low-pass Gaussian noise, usually cannot be derived in a closed form. It can be computed if the data defining the sample are available. In that case , the knowledge of the closed form validates the numerical calculation of the ensemble average of the *WD.*

II. NOTATIONS of WIGNER DISTRIBUTIONS

The Wigner distribution (*WD*) [1] of a real signal $x(t)$ may be written in the form [6]

$$W_{xx}(t,f) = 0.5W_{even} + 0.25W_{cross} = 0.25\left[W_{ss^*}(t,f) + W_{ss*}(t,-f) + W_{cross}(t,f)\right], \quad (1)$$

where $W_{ss^*}(t,f)$ is the *WD* of the analytic signal $s(t) = x(t) + j\hat{x}(t)$. Here $\hat{x}(t)$ is the Hilbert transform of $x(t)$ and $W_{ss^*}(t,-f) = W_{s^*s}(t,f)$ is the *WD* of the cojugate analytic signal $s^*(t)$. The term $W_{even}(t,f) = 0.5[W_{ss^*}(t,f) + W_{ss^*}(t,-f)]$ is called the even part. Let us remind that due to the bilinear nature, the *WD* of a real signal contains a cross term $W_{\text{cross}}(t, f)$. Boashash [7] proposed to apply $W_{ss^*}(t,f)$ instead of $W_{xx}(t,f)$ to avoid the generation of $W_{cross}(t,f)$. However, the term $W_{even}(t,f)$ does also not contain the cross-term at the cost of containing the redundant term $W_{ss^*}(t,-f)$. The cross-term $W_{cross}(t,f) = W_{cross}(t,-f)$ is also an even function of f. Therefore, all terms with the support in the half-plane $f < 0$ are redundant. In this paper, to get a more illustrative presentation, we decided to display in examples all three terms of (1). Note that it is easier to compute W_{xx} than W_{ss^*}. Some important features of the cross-terms are studied by slices of the 2-D time-frequency distributions along the line $f = 0$. Such a slice requires a two-sided representation in the frequency domain. For low-pass signals, the supports of W_{even} and W_{cross} overlap and both terms should be displayed separately. In that case, we may calculate the *WD* of the Hilbert transform $\hat{x}(t)$ of the form

$$W_{\hat{x}\hat{x}}(t,f) = 0.5W_{even} - 0.25W_{cross}. \quad (2)$$

It differs from (1) only by the sign of $W_{cross}(t,f)$. The addition of (1) and (2) yields

$$W_{even} = W_{xx}(t,f) + W_{\hat{x}\hat{x}}(t,f) \quad (3)$$

and the subtraction yields

$$W_{cross}(t,f) = 2\left[W_{xx}(t,f) - W_{\hat{x}\hat{x}}(t,f)\right]. \quad (4)$$

III. CORRELATION PRODUCTS AND FUNCTIONS

Let us define the correlation product for a single sample $s_i(t)$ of a random process $\{s(t)\}$

$$r_{ss^*}^{(i)}(t,\tau) = s_i(t+0.5\tau)s_i^*(t-0.5\tau) = x_i^+x_i^- + \hat{x}_i^+\hat{x}_i^- + j\left(x_i^+\hat{x}_i^- - \hat{x}_i^+x_i^-\right) \tag{5}$$

and the complementary correlation product [5]

$$r_{ss}^{(i)}(t,\tau) = s_i(t+0.5\tau)s_i(t-0.5\tau) = x_i^+x_i^- - \hat{x}_i^+\hat{x}_i^- + j\left(x_i^+\hat{x}_i^- + \hat{x}_i^+x_i^-\right), \tag{6}$$

where the superscript "+" denotes a function of $t + 0.5\tau$ and "-" a function of $t - 0.5\tau$. Let us denote the terms of (5) as follows

$$r_{ss^*}^{(i)}(t,\tau) = r_{xx}^{(i)} + r_{\hat{x}\hat{x}}^{(i)} + j\left(r_{x\hat{x}}^{(i)} - r_{\hat{x}x}^{(i)}\right). \tag{7}$$

Therefore, the terms of the complementary product are

$$r_{ss}^{(i)}(t,\tau) = r_{xx}^{(i)} - r_{\hat{x}\hat{x}}^{(i)} + j\left(r_{x\hat{x}}^{(i)} + r_{\hat{x}x}^{(i)}\right). \tag{8}$$

The ensemble averages of correlation products define the corresponding correlation functions. We have

$$r_{ss^*}(t,\tau) = E\left\{r_{ss^*}^{(i)}(t,\tau)\right\} = r_{xx} + r_{\hat{x}\hat{x}} + j\left[r_{x\hat{x}} - r_{\hat{x}x}\right] \tag{9}$$

and the complementary correlation function

$$r_{ss}(t,f) = E\left\{r_{ss}^{(i)}(t,\tau)\right\} = r_{xx} - r_{\hat{x}\hat{x}} + j\left[r_{x\hat{x}} + r_{\hat{x}x}\right], \tag{10}$$

where r_{xx} and $r_{\hat{x}\hat{x}}$ are autocorrelation and $r_{x\hat{x}}$ and $r_{\hat{x}x}$ cross-correlation functions.

Proper and improper processes: The authors of [5] following the ideas presented by Picinbono et all. (see reference [3]-[6] in [5]) define so-called proper and improper complex random signals. A complex zero-mean random signal $s(t)$ is called proper if $E\{s_i(t_1)s_i(t_2)\} = 0$ for all pairs (t_1, t_2). Here we have $t_1 = t + 0.5\tau$ and $t_2 = t - 0.5\tau$. The signal is proper if in (5) $r_{xx} = r_{\hat{x}\hat{x}}$ and $r_{x\hat{x}} = -r_{\hat{x}x}$. Important remark: For random signals, the correlation products defined by (5) and (6) are random functions. Differently, the ensemble averages (9) and (10) are well-defined deterministic functions [4] . Note that for proper processes, the real and imaginary terms of the complementary correlation function equal zero.

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The *WD* of a single sample of the analytic signal $s_i(t)$ defined by the Fourier transform of the correlation product (5)

$$W_{ss^*}^{(i)}(t,f)=\int_{-\infty}^{\infty} r_{ss^*}^{(i)}(t,\tau)e^{-j2\pi f\tau}d\tau \tag{11}$$

is a real function. The same formula applies for the *WD* of a real signal $x(t)$ changing the subscript ss^* to xx. Differently, the complementary *WD* defined by the Fourier transform of the complementary correlation product (6), i.e.,

$$W_{ss}^{(i)}(t,f)=\int_{-\infty}^{\infty} r_{ss}^{(i)}(t,\tau)e^{-j2\pi f\tau}d\tau \tag{12}$$

is a complex function. The insertion of (6) disregarding two vanishing integrals yields

$$W_{ss}^{(i)}(t,f)=\int_{-\infty}^{\infty}\left[r_{xx}^{(i)}(t,\tau)-r_{\hat{x}\hat{x}}^{(i)}(t,\tau)\right]\cos(2\pi f\tau)d\tau+j\int_{-\infty}^{\infty}\left[r_{x\hat{x}}^{(i)}(t,\tau)+r_{\hat{x}x}^{(i)}(t,\tau)\right]\cos(2\pi f\tau)d\tau,$$

$$W_{ss}^{(i)}(t,f)=\mathrm{Re}\left[W_{ss}^{(i)}(t,f)\right]+j\,\mathrm{Im}\left[W_{ss}^{(i)}(t,f)\right]. \tag{13}$$

The *WD* of a single sample of a random signal defined by (11) may be regarded as sample of a random proces $\left\{W_{ss^*}(t,f)\right\}$. The ensemble average is

$$T_{ss^*}(t,f)=E\left\{W_{ss^*}(t,f)\right\}=E\left\{\int_{-\infty}^{\infty} r_{ss^*}^{(i)}(t,\tau)e^{-j2\pi f\tau}d\tau\right\}. \tag{14}$$

We apply the notation *T* used in [5]. Since the operator *E* and the Fourier transform are linear, the order in (14) can be changed. We get

$$T_{ss^*}(t,f)=E\left\{W_{ss^*}^{(i)}(t,f)\right\}=\int_{-\infty}^{\infty} E\left\{r_{ss^*}^{(i)}(t,\tau)\right\}e^{-j2\pi f\tau}d\tau=\int_{-\infty}^{\infty} r_{ss^*}(t,\tau)e^{-j2\pi f\tau}d\tau. \tag{15}$$

This form shows that the ensemble average $T_{ss^*}(t.f)$ is given by the Fourier transform of the correlation function defined by (9) (see [2], Section 3.2.5). In consequence, it is also a well-defined deterministic function [4]. Computer simulations with a finite number of *N* samples confirmed that the Eq.(14) and (15) yield the same functions. Note that (15) requires the summation of *N* correlation products and calculation of a single integral while (14) requires the calculation of *N* integrals. Obviously, the computational efficiency is much better by implementation of (15) than (14).

Similarly to (15), the ensemble average of the complementary *WD* is

$$T_{ss}(t,f)=\int_{-\infty}^{\infty} r_{ss}(t,\tau)e^{-j2\pi f\tau}d\tau=\mathrm{Re}\{T_{ss}\}+j\,\mathrm{Im}\{T_{ss}\}\ . \qquad (16)$$

It can be shown that the cross-term in (1) is equal to twice the real part of the complementary *WD* defined by (13), i.e., $T_{cross}(t,f)=2\mathrm{Re}\left[T_{ss}(t,f)\right]$. Remark: In [5] the last term in Eq.(28), i.e., $\mathrm{Re}T_{ss}^{W}(t,f)$ should be multiplied by 2. Concluding, investigations about the role of the complementary *WD* coincide with investigations about the role of cross-terms and for proper processes the theoretically derived ensemble averages of cross-terms equal zero.

The nature of the terms of the ensemble average of the WD of a real signal.

The *WD* of a single sample of a real signal $x_i(t)$ is given by the Eq.(11). The corresponding ensemble average is (see (1))

$$T_{xx}(t,f)=0.5T_{even}+0.25T_{cross}=0.25\left[T_{ss^*}(t,f)+T_{s^*s}(t,f)+T_{cross}(t,f)\right] \qquad (17)$$

and of the Hilbert transform $\hat{x}_i(t)$ (see (2))

$$T_{\hat{x}\hat{x}}(t,f)=0.5T_{even}-0.25T_{cross}=0.25\left[T_{ss^*}(t,f)+T_{s^*s}(t,f)-T_{cross}(t,f)\right]. \qquad (18)$$

The energies of the signal and its Hilbert transform are equal and given by the integrals [3]

$$\text{Energy}=\int\left[x(t)\right]^2dt=\int\left[\hat{x}(t)\right]^2dt=\iint T_{xx}(t,f)\,dtdf=\iint T_{\hat{x}\hat{x}}(t,f)\,dtdf\ . \qquad (19)$$

This equation is fulfilled only if the energy of the cross term in (17) and (18) equals zero. We have

$$\iint T_{cross}(t,f)\,dtdf=0\quad . \qquad (20)$$

In many presented examples this equation is fulfilled due to the periodicity in time of $T_{cross}(t,f)$. Differently, the two terms of T_{even} are usually (but not always) unipolar functions. Both are well defined deterministic functions. Note that (19) and (20) are valid if we insert W_{xx} in place of T_{xx} [6].

V. EXAMPLES

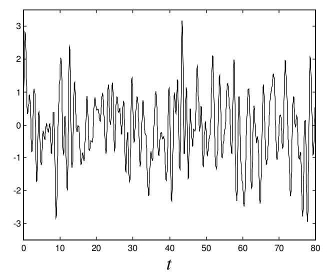
V.I. Noise signals

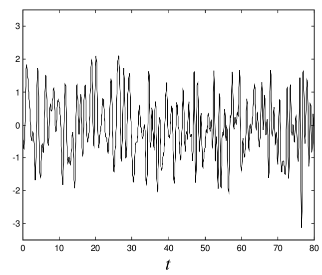
1. ***Low-pass noise***

Consider samples $x_i(t)$ of a low-pass nonstationary Gaussian noise with the power density

$$G(f) = \begin{cases} G_0 \text{ if } |f| < B(t) \\ 0.5G_0 \text{ if } |f| = B(t) \\ 0 \text{ if } |f| > B(t) \end{cases}. \tag{21}$$

If $B(t) = B$ is a constant, the noise is *WSS*. Fig.1a shows a sample of $x_i(t)$, $B = 1$, generated by the method described in [8], [9]. The method enables the generation of the Hilbert transform of $x(t)$. A sample of the nonstationary noise is shown in Fig.1b. The corresponding instantaneous bandwidth $B(t) = 1 + \left[0.25 + 0.25\tanh\left(0.16(t-40)\right)\right]$ is shown in Fig.1c.

a) 

b) 

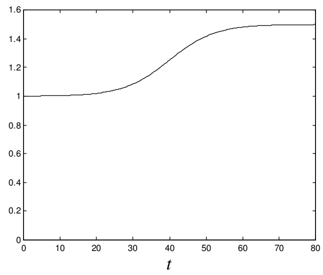
c) 

Fig. 1. a) A sample of a low-pass *WSS* Gaussian noise , b) A sample of a nonstationary noise. c) The instantaneous bandwidth $B(t) = 1 + \left[0.25 + 0.25\tanh\left(0.16(t-40)\right)\right]$ of the sample in b).

The stationary case

In the stationary case, the terms of the correlation function (9) are time independent and have the form [9]

$$r_{xx}(\tau) = r_{\hat{x}\hat{x}}(\tau) = \int_{-B}^{B} G_0 e^{-j2\pi f\tau} df = 2G_0 B \frac{\sin(2\pi B\tau)}{2\pi B\tau}, \tag{22}$$

$$r_{x\hat{x}} = -r_{\hat{x}x} = 2G_0 B \frac{1-\cos(2\pi B\tau)}{2\pi B\tau}.$$

The random process $\{x_i(t)\}$ is *WSS* and proper and the terms of the complementary correlation function (10) equal zero. Therefore, the ensemble average of cross-terms T_{cross} equals zero. In consequence, the ensemble average of the *WD* given by (17) is $T_{xx}(t,f) = 0.5T_{even} = G(f) \otimes 1_t$. The multiplication with 1_t indicates the time independence. Let us explain why having the theoretical solution it is still reasonable to calculate computer simulations.

1. The samples of noise of Fig.1 are generated by a computer using a specific algorithm. They represent a simulation of the ideal Gaussian noise. Simulations are never perfect.

2. The theoretical ensemble average is defined for an infinite number of samples. In computer simulations this number is finite. In consequence, the cancellation of the cross-term is not perfect. In the example shown below, the vanishing cross-term is modulated by a sine wave. Actually, we have not derived a theoretical explanation of this effect.

3. There are no closed forms representing the *WD*s of single samples. Therefore, comparisons of the properties of *WD*s of single samples with the ensemble averages are possible using computer simulations.

4. Computer simulations of examples with known theoretical solutions validate similar simulations for which theoretical solutions are not derived.

5. Note that computer simulations apply samples of finite length while the theoretical samples may have infinite length.

Let us present the result of computer simulations for the low-pass stationary noise. The *WD* of a single sample $x_i(t)$ defined by Eq.(1) is displayed in Fig.2. Since the supports of the two terms of $W_{xx}(t,f) = 0.5W_{even} + 0.25W_{cross}$ overlap, we display: a) W_{xx} , b) $0.5W_{even}$ and c) $0.25W_{cross}$. We observe that all terms defined by the Eq.(1) are bipolar random fields. The corresponding estimates of ensemble averages calculated using 2000 samples are shown in Fig.3 and 4. We observe that all terms of (17) are well defined deterministic functions and the random process $\{x_i(t)\}$ is *WSS* and proper. From Fig.4b and c we can see, that $T_{cross}(t,f)$ is a bipolar function due to the modulation by a sine wave of frequency $f_{cross} = 2B$. Let us remind that the zero crossing frequency of a single sample of noise equals $2B/\sqrt{3}$ (Rice formula [10]). However, the comparison of Figs.2c and 3c shows that the averaging process, as expected, almost cancels the cross-term . The slice $T_{xx}(f, t=20)$ displayed in Fig.4a (solid line) shows that the term $T_{even}(t,f)$ is also a well defined deterministic function. This slice is a good approximation of the power spectrum given by Eq.(21). The dotted line represents the vanishing cross-term. Theoretically, the power spectrum of Fig.4a should be a rectangular function.

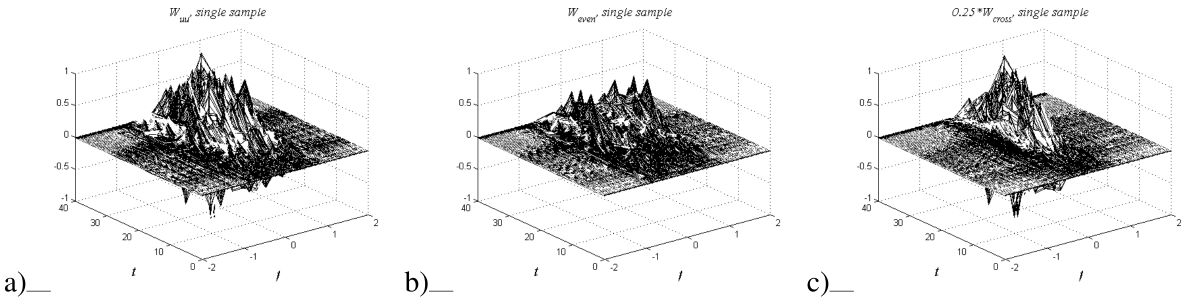


Fig.2. The *WDs* of a single sample of noise (see Fig.1a). a) $W_{xx}(t,f) = 0.5W_{\text{even}}+0.25W_{\text{cross}}$., b) $0.5W_{\text{even}}$, and c) $0.25W_{\text{cross}}$. In this example the supports of W_{even} and of W_{cross} overlap.

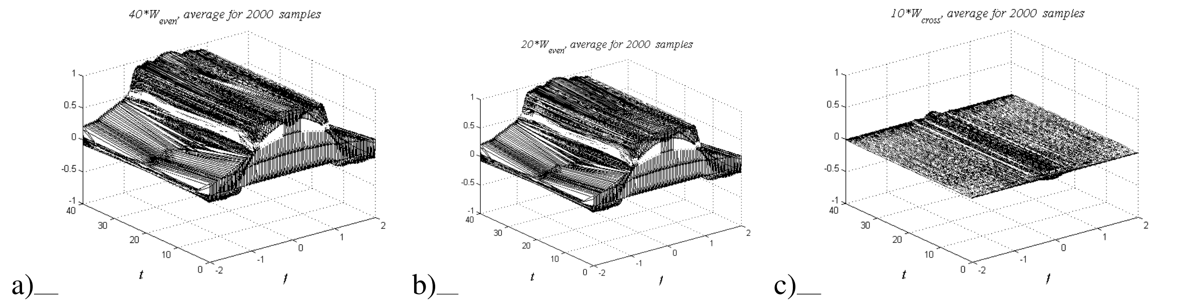


Fig.3. Estimates of ensemble averages for 2000 samples corresponding to Fig.2 :a) T_{xx}(t,f) $=0.5T_{\text{even}}+0.25T_{\text{cross}}$, b) T_{even}, and c) The vanishing term T_{ccross}.

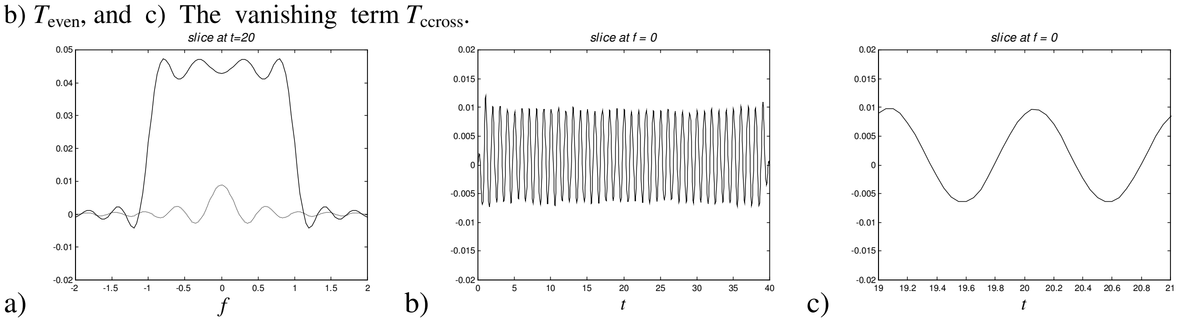


Fig.4. a) Slices of $T_{xx}\left(t=20,f\right)$. Solid line T_{even}, dotted line T_{cross}. b) The slice $T_{cross}\left(t,f=0\right)$ shows the sine wave modulating the envelope of T_{cross}, c) A fragment of the *t*-axis of b) shows a sine wave.

The nonstationary case

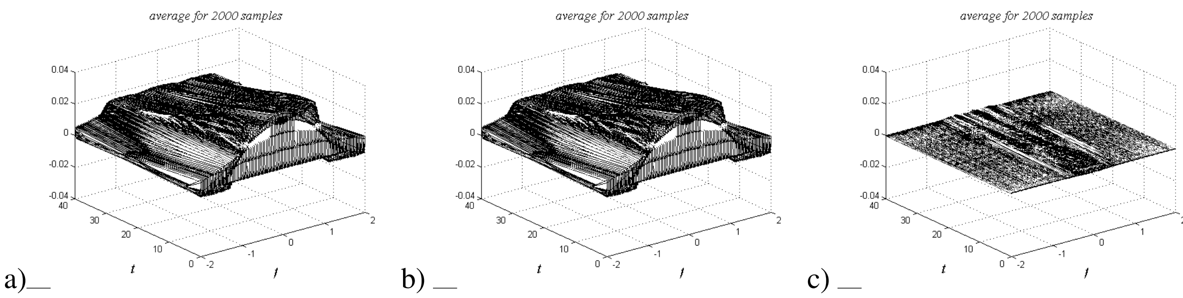


Fig.5. Nonstationary low-pass noise. The estimates of ensemble averages. a) $T_{xx}\left(t,f\right)=0.5T_{even}+0.25T_{cross}$, b) $0.5T_{even}$, c) $0.25T_{cross}$.

For the nonstationary noise with a sample function of Fig.1b, we display only the estimates of the ensemble averages calculated using 2000 samples. Fig.5 shows these averages in the same order as in

Fig.3. The cross-term in Fig.5c is negligible. A computer simulation cannot serve as an exact evidence. Neverthless, it shows that the presented nonstationary process is proper. Fig.6 shows that the vanishing cross-term is modulated by a sine wave. Detailed observations (not displayed) show that the frequency of this modulation decreases in time by about 3.8% while the bandwidth increases by 50 %. The change of the badwidth is illustrated by the slices of T_{even} displayed in Fig.7.

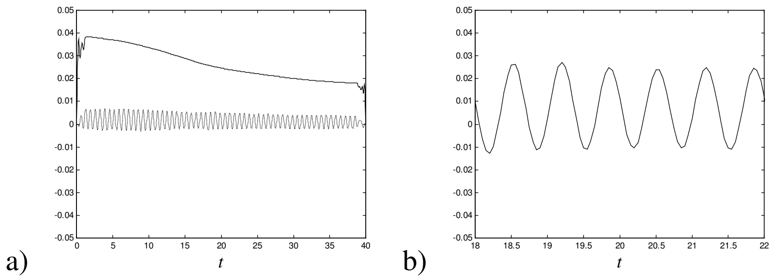


Fig.6. Nonstationary Gaussian noise. The slices of the terms of $T_{xx}\left(t, f=0\right)$. a) Solid line: $T_{even}\left(t, f=0\right)$, dotted line $T_{cross}\left(t, f=0\right)$ 2, b) A fragment for $18 < t < 22$ of $5\times T_{cross}\left(t, f=0\right)$.

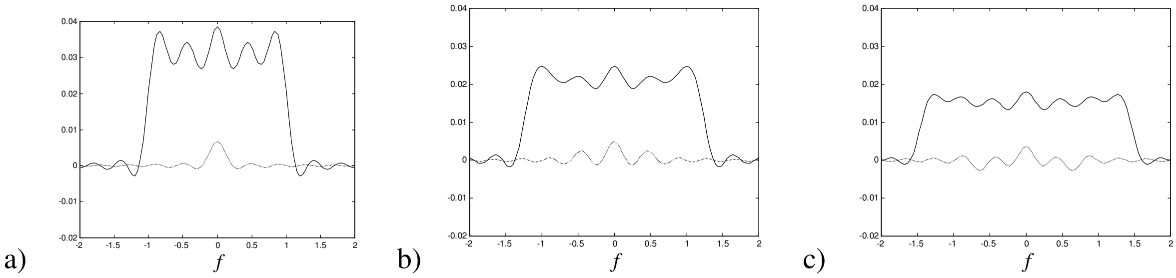


Fig.7. Nonstationary Gaussian noise. The slices $T_{xx}\left(t = const, f\right)$. a) t = 1.25, b) t = 19.85 and c) t = 35.0. Solid lines: $T_{even}\left(t = const, f\right)$ show the change of the bandwidth in time, dotted lines: negligible values of $T_{cross}\left(t = const, f\right)$.

2. *Band-pass Gaussian noise.*

The power spectrum of a band-pass Gaussian noise can be regarded as the difference of two low-pass power spectra given by (21) with $B_2 > B_1$. The corresponding time independent autocorrelation functions are

$$r_{xx}\left(\tau\right) = r_{\hat{x}\hat{x}}\left(\tau\right) = 2G_0 B_2 \frac{\sin\left(2\pi B_2 \tau\right)}{2\pi B_2 \tau} - 2G_0 B_1 \frac{\sin\left(2\pi B_1 \tau\right)}{2\pi B_1 \tau} .. \qquad (23)$$

Using (22), we get the cross-correlation functions. Of course, the band-pass Gaussian noise is proper and, as in the case of a low-pass noise, the cross-term vanish. The results of computer simulations are presented in Fig.8 and 9. In this example the supports of T_{even} and T_{cross} are disjoint.

a) b)

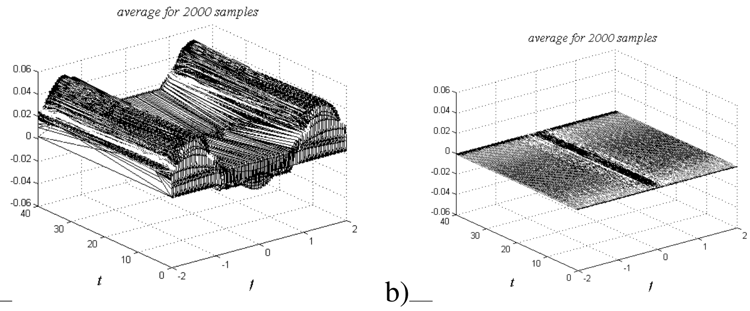


Fig.8. Band-pass noise, B_2=2, B_1 = 1. a) The ensemble average $T_{xx}(t,f)=0.5T_{even}+0.25T_{cross}$. b) The vanishing term T_{croos}.

a) b) c)

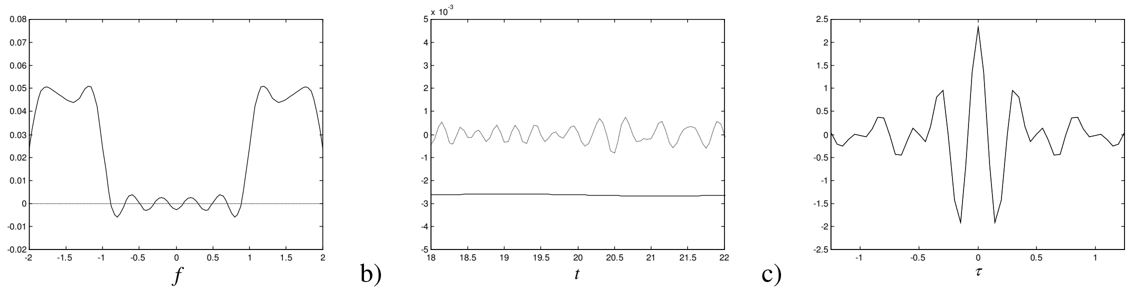


Fig.9. Band-pass noise., the slices of the *WD* of Fig.8. a) $T_{xx}(t=20,f)$, the solid line shows the estimate of the power spectrum and the dotted line the vanishing cross-term. b) A part of the t-axis of the slice $T_{cross}(t,f=0)$, average for 2000 samples (note the range of the y-axis). c) Slices of calculated correlation functions $r_{xx}(t,\tau)$ and $r_{xx}(t,\tau)$. Solid line $r_{xx}(t=20,\tau)$ and dotted line $r_{\hat{x}\hat{x}}(t=20,\tau)$. The difference is almost invisible.

The slices of calculated real parts of the correlation function defined by Eq.9, displayed in Fig.c, confirm that the $r_{xx}=r_{\hat{x}\hat{x}}$, i.e., the Gaussian band-pass noise is proper. The theoretical shape of these functions is given by the Eq.(23).

3. *Modulated low-pass noise*

A sample function of this process has the form

$$s_i(t)=y_i(t)e^{j2\pi f_c t}=y_i(t)\cos(2\pi f_c t)+jy_i(t)\sin(2\pi f_c t)=x_i(t)+j\hat{x}_i(t), \tag{24}$$

where $y_i(t)$ is a sample function of the low-pass *WSS* noise of power spectrum defined by (21).

Note that the imaginary part is the Hilbert transform of the real part only, if $f_c \geq B$, i.e., the Bedrosian's theorem can be applied [11]. The correlation functions are

$$r_{xx}(t,\tau)=\rho_{yy}(\tau)\left[\cos(2\pi f_c\tau)+\cos(4\pi f_c t)\right], \tag{25}$$

$$r_{\hat{x}\hat{x}}(t,\tau)=\rho_{yy}(\tau)\left[\cos(2\pi f_c\tau)-\cos(4\pi f_c t)\right], \tag{26}$$

where $\rho_{yy}(\tau)$ has the form (22). Evidently, $r_{xx}(t,\tau) \neq r_{\hat{x}\hat{x}}(t,\tau)$, i.e., the process is improper. The calculation of the ensemble averages of *WDs* yields

$$T_{xx}(t,f) = \int 0.5\rho_{yy}(\tau)\left[\cos(2\pi f_c \tau) + \cos 4\pi f_c t\right] e^{-j2\pi f\tau} d\tau$$

$$= 0.25[G(f-f_c) + 0.25G(f+f_c)] \otimes 1_t + 0.5G(f)\cos(4\pi f_c t), \quad (27)$$

$$T_{\hat{x}\hat{x}}(t,f) = 0.25[G(f-f_c) + 0.25G(f+f_c)] \otimes 1_t - 0.5G(f)\cos(4\pi f_c t). \quad (28)$$

Note that (27) is a specific case of (17) and (28) of (18). Therefore, the first two terms in (27) and (28) represent the unipolar term T_{even} and the third one the bipolar term Tcross. Fig.10 shows theese terms

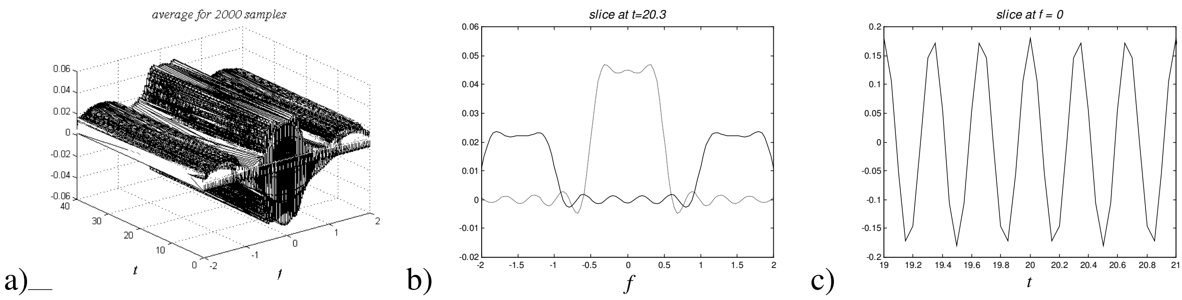


Fig.10. a) $T_{xx} = 0.5T_{even} + 0.25T_{cross}$ of modulated low-pass noise (see (24)). Note the disjoint supports of T_{even} and T_{cross}. b) The slice of the terms of T_{xx}: Solid line $0.5T_{even}(t=20,f)$, dotted line $0.25T_{cross}(t=20,f)$ c) The slice $0.25T_{cross}(t,f=0)$. The frequency $f_{\text{cross}} = 3$, i.e., equals twice of the carrier frequency $f_c = 1.5$.

calculated using the *WSS* noise of Fig.1a, $B = 0.5$ and a carrier with $f_c = 1.5$. Here, the three terms of (17) have disjoint supports. The comparison of Figs.8 and 9 with Fig.10 shows, that differently to the band-pass noise, the term T_{cross} does not vanish and is modulated by a sine wave with $f_{\text{cross}} = 2f_c = 3.0$, i.e., the same as the frequency of the cross term of the *WD* of a real signal $\cos(2\pi f_c t)$ (carrier), i.e., $\cos(4\pi f_c t)$.

***A comment to the interpretation of an example by the authors of* [5]**

The authors of [5] defined a random process with sample functions given by the real part of (24), however, with the carrier frequency $f_c = f_{c,i}$ defined as a random variable. The example presented in [5] applies $f_{c,1} = B$ and $f_{c,2} = 2B$, where B is the bandwidth of a *WSS* low-pass noise with a power spectrum given by (21). (Remark: In [5], the bandwidth defined in the text differs from the bandwidth displayed in Fig.3). Let us quote the conclusion of the authors of [5]: "Thus, $T^W_{ss^*}(t,f) =$

$\sum \Gamma_{yy}(f - f_{c,0})$ is constant in t, whereas only $T_{ss}^{W}(t,f) = \sum \Gamma_{yy}(f)\exp(j2\pi \cdot 2nf_{c,0}t)$ displays a periodic behaviour in t. Therefore, a description based on $T_{ss}^{W}(t,f)$ fails to capture the cyclo-stationary nature of x(t). Rather, it would lead us to wrongly conclude that $x(t)$ is WSS" – end of citation. Our comment is as follows: Having in mind the equiprobable values of the two carrier frequencies. the process defined in [5] may be regarded as a sum of two processes defined by (24) weighted by 0.5 . In consequence, the resulting T_{xx} is a sum of two T_{xx} given by (27) . The computer simulation for B = 0.5 is shown in Fig.11. The terms T_{even} overlap and the cross term T_{cross} has the envelope $G(f)$ modulated by the waveform of Fig.11c. Instead of the sine wave of Fig.10c, we observe a sine wave distorted by the second harmonic. The authors of [5] classify this process as cyclostationary defined by the periodicity of the cross-term w.r.t. the variable t . However, usually the process is called cyclostationary if the correlation function r(t, τ) is periodic w.r.t. the shift variable τ. Note that using the definition of cyclostationarity proposed in [5], the process defined by sample functions (24) should also be classified as cyclostationary. We are not in favour to use such a definition.

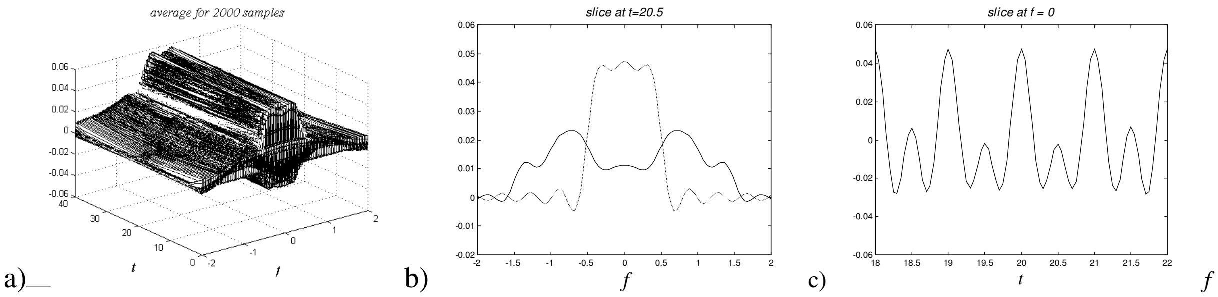


Fig.11.Illustration of the example presented in [5]. Computer simulation of T_{xx} using B = 0.5, f_{c1} = B and f_{c2} =2B. a) $T_{xx}(t,f)$, b) The slice $T_{xx}(t=20.5, f)$, solid line T_{even} , dotted line T_{cross}. c) The periodic waveform of the slice $T_{cross}(t, f=0)$.

V.2. Telecommunication Signals

All *WDs* of single samples of random telecommunication signals have the form of random bipolar fields similar as in Fig.2. Examples are here not presented [12], [13], except the case of *FSK.*

1. *Phase-Shift Keying Signals*

A sample of a radio frequency analytic signal with phase shift keing (*PSK*) is given by the formula

$$s_i(t) = A_0 e^{j[2\pi f_c t + \pi b_i(t) + \theta_i]}. \tag{29}$$

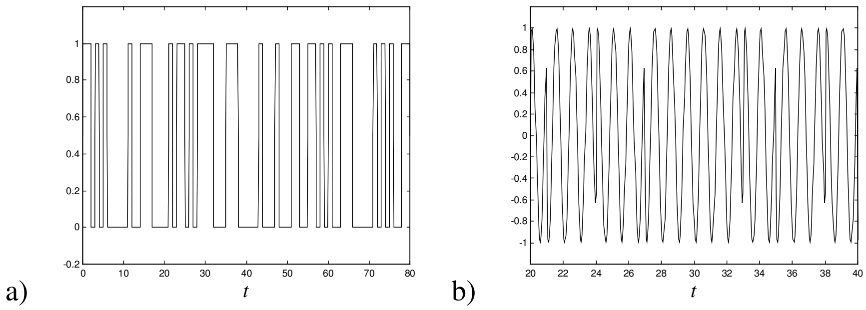


Fig.12. a) A sample of a binary random telegraph signal. b) A part of a sample of a *PSK* signal.

We assume that $b_i(t)$ is a sample of a binary random telegraph process shown in Fig.12a. It was generated using a method described in [12]. The signal is analytic for carrier frequencies sufficiently high (no leakage of the spectrum into negative frequencies [11]).

Synchroneous carrier versus asynchroneous carrier

The phase constant θ_i may be the same for all i (synchronous carrier - *SynC*) or be a random variable uniformly distributed in the interval 0 - 2π (asynchronous carrier - *AsynC*).

Synchroneous versus asynchroneous mode of the random telegraph signal

Note that the samples of the base-band random telegraph signal b_i(t) can be generated in synchronous time grid (*SynT*) or in an asynchronous time grid (*AsynT*) with transitions from the states 0 and 1 at points uniformly distributed within the elementary slot of duration *T*.

Four options of the random process with sample functions given by (25)

Therefore, there are four options in defining the random process $\{s\}$: *SynT-SynC, AsynT-SynC, SynT-AsynC* and *AsynT-AsynC*. The properties of the options are not the same. Therefore, any information about the properties of a *PSK* random process missing the information about the option is incomplete.

Derivation of the correlation functions

A sample of the correlation product is [12], [13]

$$r_{ss^*}^{(i)} = s_i\left(t+0.5\tau\right)s_i^*\left(t-0.5\tau\right) = e^{j\pi\left[b_i\left(t+0.5\tau\right)-b_i\left(t-0.5\tau\right)\right]}e^{j2\pi f_c\tau}. \quad (30)$$

The correlation function is

$$r_{ss^*}\left(t,\tau\right) = E\left\{e^{j\pi\left[b\left(t+0.5\tau\right)-b\left(t-0.5\tau\right)\right]}\right\}e^{j2\pi f_c\tau}. \quad (31)$$

Computer calculations show that $E\left\{\cos\left[\pi\left\{b(t+0.5\tau)\mp b(t-0.5\tau)\right\}\right]\right\}$ yield a waveform depending on the choice of the case *SynT or AsynT* and $E\left\{\sin\left[\pi\left\{b(t+0.5\tau)\mp b(t-0.5\tau)\right\}\right]\right\}=0$.

In the case *AsynT* we have

$$E\left\{e^{j\pi\left[b(t+0.5\tau)\mp b(t-0.5\tau)\right]}\right\}=tri(\tau/T), \tag{32}$$

where tri(τ/T) is a triangle function with a support from $-T$ to T. It yields

$$T_{ss^*}(t,f)=\int_{-T}^{T} tri(\tau/T)e^{j2\pi f_c\tau}e^{-j2\pi f\tau}d\tau=T\left[\frac{\sin\left[\pi(f-f_c)T\right]}{\pi(f-f_c)T}\right]^2\otimes 1_t\,. \tag{33}$$

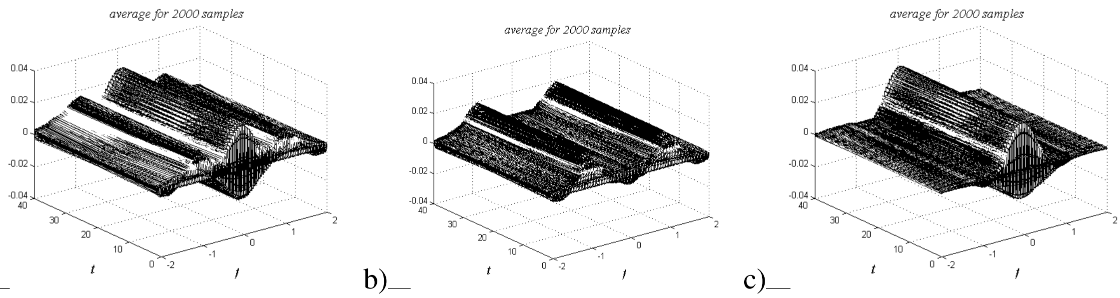
The multiplication $\otimes 1_t$ indicates the independence on time. In the case *SynT* (32) is replaced by

$$E\left\{e^{j\pi\left[b(t+0.5\tau)\pm b(t-0.5\tau)\right]}\right\}=\Pi_T(\tau/T), \tag{34}$$

where $\Pi_T(\tau/T)$ is a rectangular function with a support from $-T$ to T. It yields

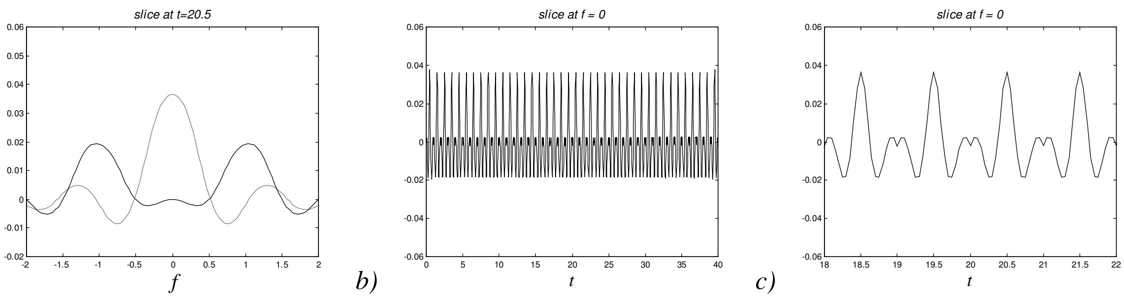
$$T_{ss^*}(t,f)=\int_{-T}^{T} e^{j2\pi f_c\tau}e^{-j2\pi f\tau}d\tau=T\left[\frac{\sin\left[2\pi(f-f_c)T\right]}{2\pi(f-f_c)T}\right]\otimes 1_t\,. \tag{35}$$

Experimental evidence that the 2PSK process, case SynT-SynC. is improper



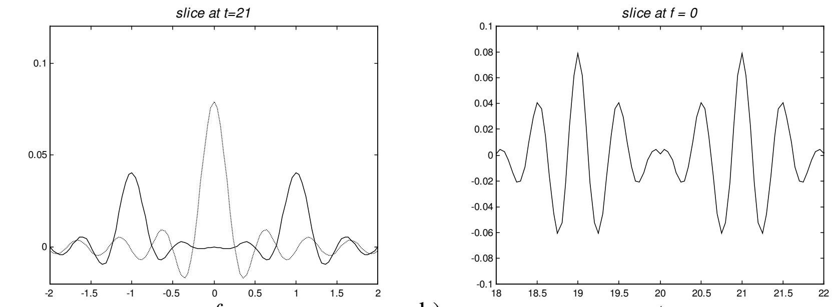
a)__ b)__ c)__

Fig.13. *WDs* of PSK signal, option *SynT-SynC* , $f_c = 1$. a) $T_{xx}(t,f)$. b) The part T_{even}, c) The part T_{ccross}.



a) *b)* *c)*

Fig.14. *PSK,* the slices of $T_{xx}(t,f)$. a) $T_{xx}(t=20.5,f)$. Solid line T_{even}, dotted line, T_{cross}, b) $T_{cross}(t,f=0)$, c) A fragment of the t-axis shows details of the waveform modulating $T_{cross}(t,f)$.



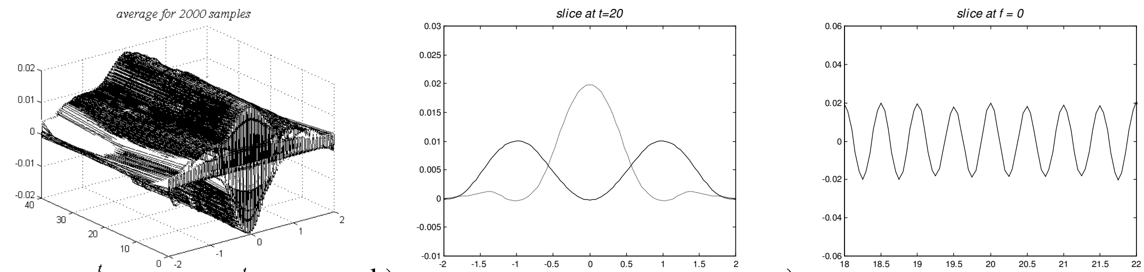
a) f b) t

Fig.15. *,PSK,* the same slices as in Fig.14a and c for a doubled value of the elementary time slot T .

Fig.13. shows the ensemble averages of the *WD*s of 2*PSK* (2 denotes binary *b*(*t*)) calculated using 2000 samples. In this example the supports of the two terms of T_{even} and of T_{cross} do not overlap. The finite value of T_{cross} (Fig.13c) shows that the 2*PSK* process, case *SynT-SynC*, is improper. The slices from Fig.14c and Fig.15b show that T_{cross} is modulated by a periodic signal . The waveform of this signal changes with the change of the elementary time slote of the transmitted random telegraph signal. *b*(*t*). Concluding, in the case *SynC-SynT* the cross-term yields the information about the carrier frequency and the length the elementary time slot (bit rate), compare Fig.14c and 15b.

Computer simulations, case* 2*PSK, **AsynT –SynC.**

The result of computer simulations is shown in Fig. 16. The comparison of Figs14 and 15 with Fig.16 shows that in the case 2*PSK,* AsynT –SynC, the waveform modulating the cross-term is a pure sine. The information about the length of the time slot *T* is lost. Note that the eventual change from the case *SynC* to *AsynC* cancells the cross-terms.



a) b) f c)

Fig.16. 2*PSK,* case AsynT-SynC. a) $T_{xx}(t,f)$, b) The slices of $T_{xx}(t=20,f)$, $T_{even}(t=20,f)$ (solid line), and $T_{cross}(t=20,f)$ (dotted line, c) The slice $T_{cross}(t,f=0)$ shows a sine waveform instead of the distorted waveforms of Fig.14 and 15. The information about the length of the time slot *T* is lost.

32*PSK,* case *SynT-SynC.*

The propertiies of ensemble averages of the *WDs* of random processes with sample functions defined by *M*-nary *PSK* differ in comparison to the binary case. Fig.17 shows the correlation functions $r_{xx}(t,\tau) \approx r_{\hat{x}\hat{x}}(t,\tau)$ calculated for 32*PSK* , case *SynT-SynC* , using 2000 samples. Evidently, their difference equals nearly zero, i.e., the process is proper. Fig.18 shows the correponding $T_{xx}(t,f) \approx T_{even}$ (the level of T_{cross} is negligible). The slices of Fig.18b) and c) show that T_{even} is unipolar and modulated by a periodic signal.

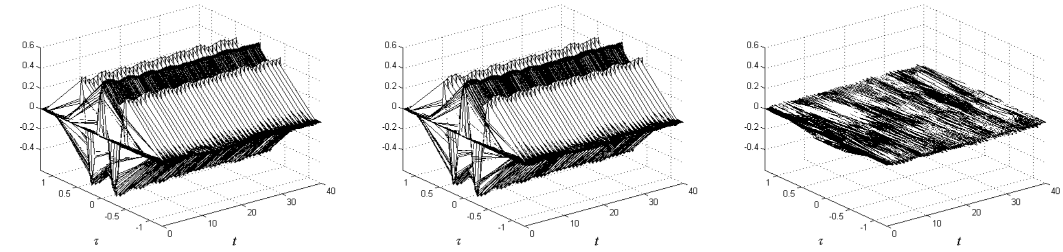
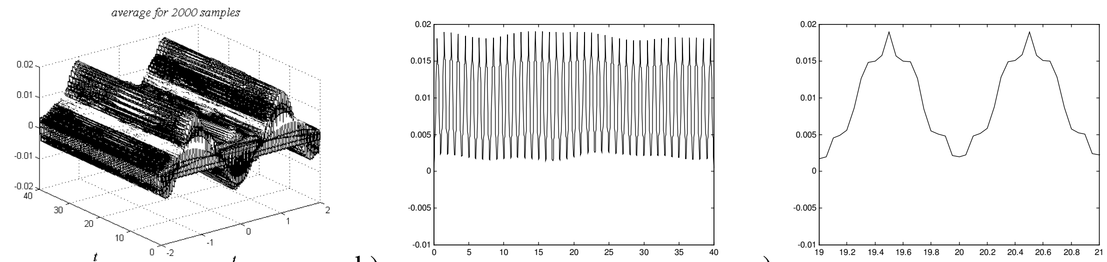


Fig.17. 32*PSK,* case *SynT-SynC*. Terms of the correlation function defined by (17). a) $r_{xx}(t,\tau)$,
b) $r_{\hat{x}\hat{x}}(t,\tau)$ and c) $r_{xx} - r_{\hat{x}\hat{x}}$. Evidently $r_{xx} \approx r_{\hat{x}\hat{x}}$ and the 32*PSK* process is proper.



a) b) c)

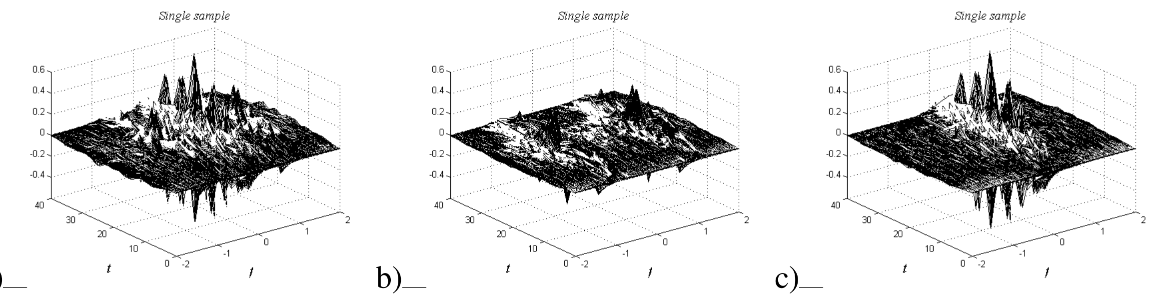
Fig.18. 32*PSK*. a) The ensemble average $T_{xx} \approx T_{even}$ (the level of T_{cross} is negligible), b) The slice $T_{even}(t, f = \pm 1)$ and c) A fragment of the *t*-axis of b).

The binary frequency shift random process is cyclostationar

A sample of the *FSK* analytic (carrier frequency f_c sufficiently large) random process is given by the formula

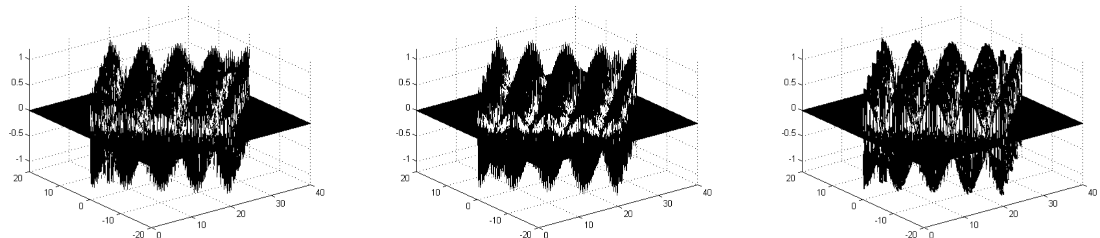
$$s_i(t) = e^{j\left[2\pi\left(f_c + \Delta f_c \times b_{1i}(t)\right) + \theta_i\right]}, \tag{36}$$

where $b_{1i} = -0.5 + b_i(t)$ yields a symmetrical frequency shift from $-0.5\Delta f_c$ to $0.5\Delta f_c$. Again, the four options defined for the *PSK* can be applied. This is a specific example. *WDs* of a single sample are shown in Fig.19. The supports of W_{even} and W_{cross} are disjoint. We observe that W_{even} is a random field. However, the random field of W_{cross} contains deterministic periodic terms. Fig.20 presents the estimates of the correlation functions defined by (17). Both terms $r_{xx}(t,\tau)$ and $r_{\hat{x}\hat{x}}(t,\tau)$ are periodic functions w.r.t. both variables, i.e., the global time t and the local time shift τ. The periodicity w.r.t. τ shows that the $2FSK$ process is cyclostationary. Fig.20c yields the evidence that $r_{xx} \neq r_{\hat{x}\hat{x}}$,i.e., the process is improper. The estimate of the ensemble average of the *WD* calculated for 1000 samples is shown in Fig.21a. The slice of the cross term in Fig.21b shows the nature of the modulation of the cross term. A periodic wave of fundamental frequency equal $2f_c$ has an envelope. The frequency of this envelope equals the double value of the frequency shift Δf_c. The slice of T_{even} in Fig.21c has the form of a Gabor wavelet. We observe here the term T_{even} is here a bipolar function of t. The width of the main period of the wavelet depends on the value of the frequency shift. Again, the change from the case *SynC* to *AsynC* cancels the cross-term.



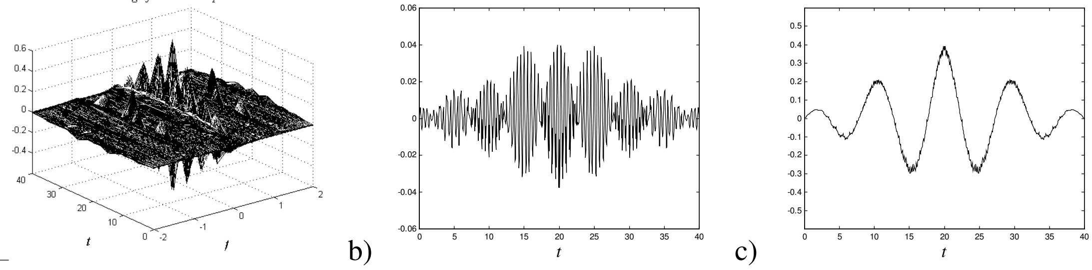
a) b) c)

Fig.19. $2FSK$, case *SynT-SynC, WDs* of a single sample, carrier frequency $f_c = 1$, frequency deviation $\Delta f_c = 0.1$. a) W_{xx} , b) W_{even} and c) W_{cross}.



a) b) c)

Fig.20. $2FSK$, case *SynT-SynC*. Terms of the correlation function defined by (17). a) $r_{xx}(t,\tau)$, b) $r_{\hat{x}\hat{x}}(t,\tau)$ and c) $r_{xx} - r_{\hat{x}\hat{x}}$. Evidently $r_{xx} \neq r_{\hat{x}\hat{x}}$ are periodic functions w.r.t. t and τ.



a) b) c)

Fig.21. 2*FSK,* case *SynT-SynC,* f_c =1, Δf_c = 0... a) The estimate of the ensemble average $T_{xx}(t,f) = 0.5T_{even} + 0.25T_{cross}$, b) The slice $T_{cross}(t, f = 0)$ and c) The slice $T_{even}(t, f = \pm 1)$.

The M-nary PSK and FSK have equal ensemble averages of the Wigner distributions and are proper.

However, only the 2*FSK* process is improper. For the M-*FSK* process, M ≥ 8, $r_{xx}(t,\tau) \approx r_{\bar{x}\bar{x}}(t,\tau)$ and both are functions only of τ. Therefore, the process is proper and the cross-term vanish. We found that the images in Fig.18 and 19 calculated for 32*PSK* are practically the same for 32*FSK.*

CONCLUSIONS

1. We confirmed in several examples that *WDs* of a single sample of a random process have the form of a bipolar random field and that their ensemble averages are well defined deterministic functions.

2. From the point of view of illustrative presentation and computational efficiency it was convenient to calculate the *WDs* of real signals in the form $W_{xx}(t,f) = 0.5W_{even} + 0.25W_{cross}$ or for ensemble averages $T_{xx}(t,f) = 0.5T_{even} + 0.25T_{cross}$ having in mind that the parts in the half-plane $f < 0$ are redundant. This representation enables the presentation of important slices $T_{cross}(t, f = 0)$. If the supports of W_{even} and W_{cross} overlap, they can be calculated separately using (3) and (4).

3. The equation (20) shows that the energy of cross-terms equals zero.

4. The double value of the real part of the complementary *WD* defined in [5] coincides with the cross-term T_{cross}. In consequence, any statements about the role of the complementary *WD* apply for the statements about the role of cross-terms.

5. The 2*PSK* and 2*FSK* processes , cases *SynT-SynC* and *AsynT-Sync,* are improper. Differently, the corresponding *M-PSK* and *M-FSK* prosesses, $M > 8$, are proper (the cross-terms vanish). Computer calculations show, that for $M = 32$, the ensemble averages of *WDs* of *PSK* and *FSK* processes are almost the same.

6. For many processes, for example stationary and nonstationary Gaussian noise and all processes with harmonic carriers of uniformly distributed random phase, the cross-terms vanish. In computer simulations, the level of these cross-terms decreases with increasing number of samples.

7. For selected processes, the ensemble average of the cross-terms contain information about the features of the process not included in the even part. For example, for 2*PSK*, option *SynT*-SynC, the cross-term yield the information about the length of the elementary time-slot of the random telegrraph signal. In the case 2*FSK,* option *SynT*-SynC, the cross-terms yield the information about the value of the frequency shift. However, the same information is included in the even part.

8. Having in mind the points 5, 6 and 7, we believe that the conclusion of the authors of [5] that the complementary *WD* matters in stochastic time-frequency analysis using the analytic signal should be replaced by the statement in which cases it matters.

9. Due to the large number of samples required to calculate the estimate of ensemble averages, in all cases when the teorethical model based on experimental data cannot be developed, the calculation of ensemble averages is impossible.

10. Let us have a following comment about the role of cross-terms. The instantaneous frequency of a single sample of the random process is given in terms of the *WD* by the formula

$$f(t) = \frac{\int_{-\infty}^{\infty} f W_{ss^*}(t,f)\,df}{\int_{-\infty}^{\infty} W_{ss^*}(t,f)\,df}. \tag{37}$$

The instantaneous phase is given by the integral

$$\varphi(t) = \int f(t)\,dt + \varphi_0, \tag{38}$$

where φ_0 is the integration constant. The information about φ_0 is not included in $W_{ss^*}(t,f)$. In the case of the simple harmonic signal $x(t) = \cos(2\pi f_c t + \varphi_o)$, it may be recovered from the cross-term $2\cos(4\pi f_c t + \varphi_0)$. However, $x(t) = \cos(2\pi f_c t + \varphi_o)$ defines a random process only, if φ_0 is a random variable. And if this variable is uniformely distributed in the interval 0 -2π, the cross term vanishes. Note that the ensemble average of the instantaneous frequency can not be calculated by inserting in (37) T_{ss^*} instead of W_{ss^*}. We believe that the calculation of ensemble averages of instantaneous frequency has no sense.

Concluding remark: Investigations of ensemble averages of *WD*s of random processes give a deeper insight to the properties of the processes. However, their significance for practical applications could be questioned.

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