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Various Approaches to the Theory of Complex and Hypercomplex Analytic Signals

Stefan L. Hahn and Kajetana M. Snopek

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Institute of Radioelectronics, Nowowiejska 15/19, 00-665 Warsaw, tel./fax: (4822) 8255248

e-mail: S.Hahn@ire.pw.edu.pl. K.Snopek@ire.pw.edu.pl

Introduction

The 2-D complex analytic signals with single-quadrant spectra have been defined in [1] and later presented in [2]. Later, a similar hypercomplex quaternionic analytic signal has been defined [3]. Both approaches use so called Hilbert quadruples. The author of [1] defined three Hilbert transforms of 2-D functions: two partial transforms of a real function $u(x_2, x_1)$ denoted $v_1(x_2,x_1)=\mathrm{H}_1\{u(x_2,x_1)\}$ and $v_2(x_2,x_1)=\mathrm{H}_2\{u(x_2,x_1)\}$ and a total transform $v(x_2,x_1)=\mathrm{H}\{u(x_2,x_1)\}$. This paper shows that complex and hypercomplex approaches are equivalent and in complex/hypercomplex signals the same Hilbert transforms are used. We consider 2-D analytic signals with a real part given by

$$u(x_2,x_1)=u_{ee}(x_2,x_1)+u_{eo}(x_2,x_1)+u_{oe}(x_2,x_1)+u_{oo}(x_2,x_1), \tag{1}$$

i.e., a union of even-even (*ee*), even-odd (*eo*), odd-even (*oe*) and odd-odd (*oo*) parts [2] as follows

$$u_{ee}(x_2,x_1)=\frac{u(x_2,x_1)+u(x_2,-x_1)+u(-x_2,x_1)+u(-x_2,-x_1)}{4}, \tag{2}$$

$$u_{eo}(x_2,x_1)=\frac{u(x_2,x_1)-u(x_2,-x_1)+u(-x_2,x_1)-u(-x_2,-x_1)}{4}, \tag{3}$$

$$u_{oe}(x_2,x_1)=\frac{u(x_2,x_1)+u(x_2,-x_1)-u(-x_2,x_1)-u(-x_2,-x_1)}{4}, \tag{4}$$

$$u_{oo}(x_2,x_1)=\frac{u(x_2,x_1)-u(x_2,-x_1)-u(-x_2,x_1)+u(-x_2,-x_1)}{4}. \tag{5}$$

Notice that in (1)-(5) we used the order of subscripts (x_2, x_1) (instead of (x_1, x_2)). It is due to the notation introduced in [1], [2], where e is a binary number 0 and o is a binary 1. It means that $u_{eo}(x_2,x_1)$ is an even function w.r.t. x_2 and an odd function w.r.t. x_1. The signs of terms in nominators of (2)-(5) are equal to products of odd-indexed variables.

A. 2-D Fourier transform of a 2-D real signal

The 2-D Fourier transform (2-D FT) of (1) has the form (we use the imaginary unit e_1 in the exponent):

$$U(f_2,f_1)=\iint_{\mathbb{R}^2} u(x_2,x_1)e^{-e_1\alpha_1}e^{-e_1\alpha_2}dx_2dx_1 \tag{6}$$

where $\alpha_1 = 2\pi f_1 x_1$ and $\alpha_2 = 2\pi f_2 x_2$. The insertion of (1) into (6) yields the spectrum in the form of a complex sum of four terms:

$$U(f_2, f_1) = U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe}) \tag{7}$$

where

$$U_{ee}(f_2, f_1) = \iint_{\mathbb{R}^2} u_{ee}(x_2, x_1)\cos\alpha_2 \cos\alpha_1 dx_2 dx_1 \,, \tag{8}$$

$$U_{eo}(f_2, f_1) = \iint_{\mathbb{R}^2} u_{eo}(x_2, x_1)\cos\alpha_2 \sin\alpha_1 dx_2 dx_1 \,, \tag{9}$$

$$U_{oe}(f_2, f_1) = \iint_{\mathbb{R}^2} u_{oe}(x_2, x_1)\sin\alpha_2 \cos\alpha_1 dx_2 dx_1 \,, \tag{10}$$

$$U_{oo}(f_2, f_1) = \iint_{\mathbb{R}^2} u_{oo}(x_2, x_1)\sin\alpha_2 \sin\alpha_1 dx_2 dx_1 \,. \tag{11}$$

B. The Quaternionic Fourier transform of a 2-D real signal

There are various definitions of the quaternionic Fourier transform (QFT). Some authors [3], [4] use the two-sided form of the QFT introduced by Ell [5]:

$$QFT(f_2, f_1) = U_q(f_2, f_1) = \iint_{\mathbb{R}^2} e^{-e_1\alpha_1} u(x_2, x_1) e^{-e_2\alpha_2} dx_2 dx_1 \tag{12}$$

where e_1 and e_2 are imaginary units of the algebra of quaternions $\mathbb{H}$ with a basis $\{e_1, e_2, e_3\}$ (the basic properties of quaternions are presented in Appendix A). The insertion of (1) into (12) yields

$$U_q(f_2, f_1) = U_{ee} - e_1 U_{eo} - e_2 U_{oe} + e_3 U_{oo} \tag{13}$$

where all terms are given by (8)-(11) and according to the quaternionic algebra multiplication rules: $e_3 = e_1 e_2$ (see Appendix A). There exists a formula relating the two-sided QFT and the standard FT of a real signal $u(x_2, x_1)$ derived by Pei *et al.* [6] and given by

$$U_q(f_2, f_1) = U(f_2, f_1)\frac{1-e_3}{2} + U(-f_2, f_1)\frac{1+e_3}{2} \,. \tag{14}$$

The complex analytic signal with a single-quadrant spectrum

In this section, we briefly recall the basic elements of the theory of 2-D analytic signals with a single-quadrant spectra. We are focused on the complex analytic signal $\Psi_1(x_2, x_1)$

with a spectrum in the first quadrant of the frequency space [1], [2] which is defined by the inverse Fourier transform

$$\Psi_1(x_2,x_1)=\iint_{\mathbb{R}^2}(1+\operatorname{sgn} f_1)(1+\operatorname{sgn} f_2)U(f_2,f_1)e^{e_1\alpha_1}e^{e_1\alpha_2}df_2df_1\ . \tag{15}$$

In the signal domain (x_2, x_1), the definition (15) corresponds to the convolution

$$\Psi_1(x_2,x_1)=u(x_2,x_1)**\left[\delta(x_1)+e_1\frac{1}{\pi x_1}\right]\left[\delta(x_2)+e_1\frac{1}{\pi x_2}\right]. \tag{16}$$

We get

$$\Psi_1(x_2,x_1)=u(x_2,x_1)-v(x_2,x_1)+e_1\left[v_1(x_2,x_1)+v_2(x_2,x_1)\right] \tag{17}$$

where

$$v_1(x_2,x_1)=-e_1\iint_{\mathbb{R}^2}\operatorname{sgn}(f_1)U(f_2,f_1)e^{e_1(\alpha_1+\alpha_2)}df_2df_1\ , \tag{18}$$

$$v_2(x_2,x_1)=-e_1\iint_{\mathbb{R}^2}\operatorname{sgn}(f_2)U(f_2,f_1)e^{e_1(\alpha_1+\alpha_2)}df_2df_1 \tag{19}$$

are called *partial Hilbert transforms* w.r.t. a single variable x_1 or x_2 respectively, and

$$v(x_2,x_1)=-\iint_{\mathbb{R}^2}\operatorname{sgn} f_1\operatorname{sgn} f_2U(f_2,f_1)e^{e_1(\alpha_1+\alpha_2)}df_2df_1 \tag{20}$$

is a *total Hilbert transform* w.r.t. (x_2, x_1).

A. Hilbert transforms in the complex case

Now, let us derive total and partial Hilbert transforms in the form of a sum of their even-even, even-odd, odd-even and odd-odd parts. We insert the Fourier transform $U(f_2, f_1)$ given by (7) into (18)-(20) and yield the following forms of the Hilbert transforms (the full derivation is presented in Appendix B):

$$v_1(x_2,x_1)=-v_{1_{ee}}^{(eo)}+v_{1_{eo}}^{(ee)}-v_{1_{oe}}^{(oo)}+v_{1_{oo}}^{(oe)}, \tag{21}$$

$$v_2(x_2,x_1)=-v_{2_{ee}}^{(oe)}-v_{2_{eo}}^{(oo)}+v_{2_{oe}}^{(ee)}+v_{2_{oo}}^{(eo)}, \tag{22}$$

$$v(x_2,x_1)=v_{ee}^{(oo)}-v_{eo}^{(oe)}-v_{oe}^{(eo)}+v_{oo}^{(ee)}\ . \tag{23}$$

Let us note that in (21)-(23) upper subscripts indicate the even/odd parity of the corresponding term of $U(f_2, f_1)$ given by (7) and the lower subscripts – the even/odd parity of the corresponding Hilbert transform.

The hypercomplex analytic signal with a single-quadrant spectrum

Again, let us recall the elements of the theory of the hypercomplex analytic signal with a single-quadrant spectrum defined in [3] using the inverse Quaternionic Fourier Transform (QFT) of a single-quadrant quaternionic spectrum, i.e.:

$$\Psi_q(x_1, x_2) = \mathrm{QFT}^{-1}\left\{\left(1+\mathrm{sgn}(f_1)\right)\left(1+\mathrm{sgn}(f_2)\right)U_q(f_2, f_1)\right\}, \tag{24}$$

i.e., using the same single-quadrant operator as in (15). Using the definition of Ell [4] of the inverse QFT we have

$$\Psi_q(x_2, x_1) = \iint_{\mathbb{R}^2} e^{e_1\alpha_1}\left(1+\mathrm{sgn}\ f_1\right)\left(1+\mathrm{sgn}\ f_2\right)U_q(f_2, f_1)e^{e_2\alpha_2}df_2df_1\,. \tag{25}$$

Let us remark that there exists an alternative definition of the inverse QFT (called Right-side QFT) introduced by Ell [4] and also used by Hitzer [5]. It differs from (25) by the order of terms under the integral. In this case we have

$$\Psi_q(x_2, x_1) = \iint_{\mathbb{R}^2} \left(1+\mathrm{sgn}\ f_1\right)\left(1+\mathrm{sgn}\ f_2\right)U_q(f_2, f_1)e^{e_2\alpha_2}e^{e_1\alpha_1}df_2df_1\,. \tag{26}$$

Let us note the reversed order of indexes in exponents in (26). In our investigations we use the definition (25) which corresponds in the signal domain to the convolution of the 2-D real signal $u(x_1, x_2)$ with the 2-D hypercomplex delta distribution [6]:

$$\Psi_q(x_2, x_1) = u(x_2, x_1) ** \left(\delta(x_1) + e_1\frac{1}{\pi x_1}\right)\left(\delta(x_2) + e_2\frac{1}{\pi x_2}\right). \tag{27}$$

We have

$$\begin{aligned}\Psi_q(x_2, x_1) &= u(x_2, x_1) ** \left(\delta(x_1)\delta(x_2) + \delta(x_2)\frac{e_1}{\pi x_1} + \delta(x_1)\frac{e_2}{\pi x_2} + e_3\frac{1}{\pi^2 x_1 x_2}\right)\\ &= u ** \delta(x_1)\delta(x_2) + e_1\cdot\left[u ** \frac{\delta(x_2)}{\pi x_1}\right] + e_2\cdot\left[u ** \frac{\delta(x_1)}{\pi x_1}\right] + e_3\cdot\left[u ** \frac{1}{\pi^2 x_1 x_2}\right]\end{aligned} \tag{28}$$

and finally

$$\Psi_q(x_2, x_1) = u(x_2, x_1) + e_1 v_1(x_2, x_1) + e_2 v_2(x_2, x_1) + e_3 v(x_2, x_1)\,. \tag{29}$$

A. Hilbert transforms in the hypercomplex case

Our goal is to derive the total and partial Hilbert transforms (similarly to (21)-(23)) being the terms of the quaternionic signal (29). We insert the quaternionic spectrum $U_q(f_2, f_1)$ given by (13) into (25) (the full derivation is given in Appendix C) and get the four terms of (29)

$$v_1(x_2, x_1) = -v_{1_{ee}}^{(eo)} + v_{1_{eo}}^{(ee)} - v_{1_{oe}}^{(oo)} + v_{1_{oo}}^{(oe)}, \tag{30}$$

$$v_2(x_2, x_1) = -v_{2_{ee}}^{(oe)} - v_{2_{eo}}^{(oo)} + v_{2_{oe}}^{(ee)} + v_{2_{oo}}^{(eo)}, \tag{31}$$

$$v(x_2, x_1) = v_{ee}^{(oo)} - v_{eo}^{(oe)} - v_{oe}^{(eo)} + v_{oo}^{(ee)}. \tag{32}$$

We observe that (30)-(31) have the same form as (21)-(23).

Appendix A

Algebra of quaternions

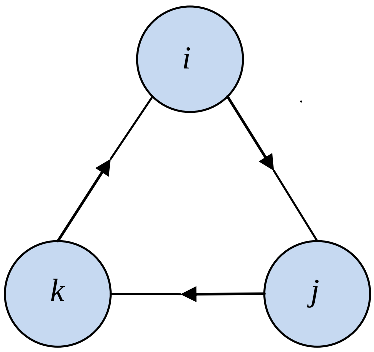
The quaternions form a non-commutative algebra of order 4 over $\mathbb{R}$,denoted with $\mathbb{H}$. Its elements q are ordered pairs of complex numbers:

$$q = (z_0, z_1) = ((r_0, r_1), (r_2, r_3));\ z_0, z_1 \in \mathbb{C},\ r_0, r_1, r_2, r_3 \in \mathbb{R}:$$

$$q = z_0 + z_1 \cdot j = (r_0 + r_1 \cdot i) + (r_2 + r_3 \cdot i) \cdot j \tag{A1}$$

Applying the Hamilton's multiplication rules [ ] of imaginary units in $\mathbb{H}$ (see Fig. A1), we obtain

$$q = z_0 + z_1 \cdot j = r_0 + r_1 \cdot i + r_2 \cdot j + r_3 \cdot k\,. \tag{A2}$$



a)

Table A1

×	***1***	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$
1	1	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	-1	e_3	$-e_2$
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	-1

b)

Fig. A1 Multiplication of imaginary units in $\mathbb{H}$: a) using the Hamilton's multiplication rule, b) using the table of multiplication

In Fig. A1 a, three successive imaginary units determine one rule of multiplication. The first two units are respectively a multiplicand and a multiplier and the third one - the product.

If we move clockwise, we have for example, $i \cdot j = k$ and $k \cdot i = j$. If we multiply the units counterclockwise, the product gets the minus sign. It means that the multiplication in $\mathbb{H}$ is not commutative and we have $j \cdot i = -j \cdot i = -k$ and $i \cdot k = -k \cdot i = -j$.

Some authors apply another notation of imaginary units (, i.e., they replace *i*, *j*, *k* with e_1, e_2, e_3 (see Table A1). In algebras of higher orders such a notation is more convenient due to the large number of imaginary units. So, the general form of a quaternion (A2) is

$$q = r_0 + r_1 \cdot e_1 + r_2 \cdot e_2 + r_3 \cdot e_3. \tag{A3}$$

The conjugate of ( ) is

$$q^* = r_0 - r_1 \cdot e_1 - r_2 \cdot e_2 - r_3 \cdot e_3 \tag{A4}$$

and its norm is

$$|q| = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}\,. \tag{A5}$$

The quaternions (A3) with $r_0 = 0$ are refered as *pure* quaternions and those with $|q| = 1$ as *unit* (*unitary*) quaternions.

Appendix B

Derivation of Hilbert transforms of a 2-D complex signal with the first-quadrant spectrum

In this part, we derive the Hilbert transforms (partial and total) starting with the decomposition of the spectrum of the real signal $u(x_2, x_1)$ into its even-even, even-odd, odd-even and odd-odd parts. It is known [1], [2] that the partial Hilbert transform $v_1(x_2, x_1)$ is given by (18). Introducing the Fourier spectrum of a real signal $u\,(x_2, x_1)$ given by (7) into (18), we have

$$\begin{aligned}
&v_1(x_2, x_1) \\
&\quad = -e_1 \iint_{\mathbb{R}^2} \operatorname{sgn}(f_1)\left[U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe})\right] e^{e_1(\alpha_1+\alpha_2)} df_2 df_1 = \\
&\quad = -e_1 \iint_{\mathbb{R}^2} \operatorname{sgn}(f_1)\left[U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe})\right]\left[\cos(\alpha_1+\alpha_2) + e_1 \sin(\alpha_1+\alpha_2)\right] df_2 df_1 = \\
&\quad = \iint_{\mathbb{R}^2} \operatorname{sgn}(f_1) U_{ee} \cos\alpha_2 \sin\alpha_1 df_2 df_1 - \iint_{\mathbb{R}^2} \operatorname{sgn}(f_1) U_{oo} \sin\alpha_2 \cos\alpha_1 df_2 df_1 - \\
&\qquad - \iint_{\mathbb{R}^2} \operatorname{sgn}(f_1) U_{eo} \cos\alpha_2 \cos\alpha_1 df_2 df_1 + \iint_{\mathbb{R}^2} \operatorname{sgn}(f_1) U_{oe} \sin\alpha_2 \sin\alpha_1 df_2 df_1.
\end{aligned} \tag{B1}$$

We apply the following notation: upper subscripts indicate the even/odd parity of the corresponding term of $U(f_2, f_1)$ given by (7) and the lower subscripts – the even/odd parity of the corresponding Hilbert transform. We obtain after reordering the terms

$$v_1(x_2, x_1) = -v_{1_{ee}}^{(eo)} + v_{1_{eo}}^{(ee)} - v_{1_{oe}}^{(oo)} + v_{1_{oo}}^{(oe)}. \tag{B2}$$

Analogously, the partial Hilbert transform $v_2(x_2, x_1)$ [1], [2] is given by (19). Repeating the same procedure as above, we get

$$\begin{aligned} v_2(x_2,x_1) & \\ &= -e_1 \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2)\left[U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe})\right]\left[\cos(\alpha_1+\alpha_2) + e_1 \sin(\alpha_1+\alpha_2)\right] df_2 df_1 = \\ &= \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2) U_{ee} \sin\alpha_2 \cos\alpha_1 df_2 df_1 - \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2) U_{oo} \cos\alpha_2 \sin\alpha_1 df_2 df_1 - \\ &\quad + \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2) U_{eo} \cos\alpha_2 \cos\alpha_1 df_2 df_1 - \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2) U_{oe} \sin\alpha_2 \sin\alpha_1 df_2 df_1 \end{aligned} \tag{B3}$$

and finally

$$v_2(x_2, x_1) = v_{2_{oe}}^{(ee)} + v_{2_{oo}}^{(eo)} - v_{2_{ee}}^{(oe)} - v_{2_{eo}}^{(oo)}. \tag{B4}$$

Analogously, the total Hilbert transform (20) is given by

$$\begin{aligned} v(x_2,x_1) & \\ &= -\iint_{\mathbb{R}^2} \operatorname{sgn}(f_2)\operatorname{sgn}(f_1)\left[U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe})\right]\left[\cos(\alpha_1+\alpha_2) + e_1 \sin(\alpha_1+\alpha_2)\right] df_2 df_1 = \\ &= \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2)\operatorname{sgn}(f_1) U_{ee} \sin\alpha_2 \sin\alpha_1 df_2 df_1 + \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2)\operatorname{sgn}(f_1) U_{oo} \cos\alpha_2 \cos\alpha_1 df_2 df_1 - \\ &\quad - \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2)\operatorname{sgn}(f_1) U_{eo} \sin\alpha_2 \cos\alpha_1 df_2 df_1 - \iint_{\mathbb{R}^2} \operatorname{sgn}(f_2)\operatorname{sgn}(f_1) U_{oe} \cos\alpha_2 \sin\alpha_1 df_2 df_1. \end{aligned} \tag{B8}$$

We obtain

$$v(x_2, x_1) = v_{oo}^{(ee)} + v_{ee}^{(oo)} - v_{oe}^{(eo)} - v_{eo}^{(oe)} \tag{B9}$$

Appendix C

Derivation of Hilbert transforms of a 2-D hypercomplex signal with the first-quadrant spectrum

Let us derive the four terms of the hypercomplex analytic signal given by (29). Inserting into (29) the quaternionic spectrum U_q given by (13) we get

$$\begin{aligned}\Psi_q(x_2,x_1) &= \iint_{\mathbb{R}^2} e^{e_1\alpha_1}\mathbf{1}(f_2,f_1)U_q(f_2,f_1)e^{e_2\alpha_2}df_2df_1\\ &= \iint_{\mathbb{R}^2} e^{e_1\alpha_1}\mathbf{1}(f_2,f_1)(U_{ee}-e_1U_{eo}-e_2U_{oe}+e_3U_{oo})e^{e_2\alpha_2}df_2df_1 \\ &= \iint_{\mathbb{R}^2}(\cos\alpha_1+e_1\sin\alpha_1)\mathbf{1}(f_2,f_1)(U_{ee}-e_1U_{eo}-e_2U_{oe}+e_3U_{oo})(\cos\alpha_2+e_2\sin\alpha_2)df_2df_1\end{aligned} \tag{C1}$$

where $\mathbf{1}(f_2,f_1)=(1+\operatorname{sgn} f_1)(1+\operatorname{sgn} f_2)$. The real part of (C1) is

$$\begin{aligned}u(x_2,x_1) &= \operatorname{Re}\{\Psi_q(x_2,x_1)\} = \\ &= \iint_{\mathbb{R}^2}(U_{ee}\cos\alpha_1\cos\alpha_2+U_{eo}\cos\alpha_2\sin\alpha_1+U_{oe}\sin\alpha_2\cos\alpha_1+U_{oo}\sin\alpha_1\sin\alpha_2)df_2df_1 \\ &= u_{ee}(x_2,x_1)+u_{eo}(x_2,x_1)+u_{oe}(x_2,x_1)+u_{oo}(x_2,x_1).\end{aligned} \tag{C2}$$

The partial Hilbert transform $v_1(x_2, x_1)$ is given by the formula:

$$\begin{aligned}&v_1(x_2,x_1)\\ &= \iint_{\mathbb{R}^2}\operatorname{sgn} f_1(U_{ee}\cos\alpha_2\sin\alpha_1-U_{eo}\cos\alpha_2\cos\alpha_1+U_{oe}\sin\alpha_2\sin\alpha_1-U_{oo}\sin\alpha_2\cos\alpha_1)df_2df_1\\ &= v_{1_{eo}}^{(ee)}-v_{1_{ee}}^{(eo)}+v_{1_{oo}}^{(oe)}-v_{1_{oe}}^{(oo)}.\end{aligned} \tag{C3}$$

Similarly, we get the formula defining the partial Hilbert transform $v_2(x_2, x_1)$ in the hypercomplex case:

$$\begin{aligned}&v_2(x_2,x_1)\\ &= \iint_{\mathbb{R}^2}\operatorname{sgn} f_2(U_{ee}\sin\alpha_2\cos\alpha_1+U_{eo}\sin\alpha_2\sin\alpha_1-U_{oe}\cos\alpha_2\cos\alpha_1-U_{oo}\cos\alpha_2\sin\alpha_1)df_2df_1\\ &= v_{2_{oe}}^{(ee)}+v_{2_{oo}}^{(eo)}-v_{2_{ee}}^{(oe)}-v_{2_{eo}}^{(oo)}.\end{aligned} \tag{C4}$$

The total Hilbert transform $v(x_2, x_1)$ is given by

$$\begin{aligned}v(x_2,x_1) &= \iint_{\mathbb{R}^2}\operatorname{sgn}(f_2)\operatorname{sgn}(f_1)\cdot\\ &\cdot(U_{ee}\sin\alpha_2\sin\alpha_1-U_{eo}\cos\alpha_2\sin\alpha_1-U_{oe}\cos\alpha_2\sin\alpha_1+U_{oo}\cos\alpha_2\cos\alpha_1)df_2df_1\\ &= v_{oo}^{(ee)}-v_{oe}^{(eo)}-v_{eo}^{(oe)}+v_{ee}^{(oo)}.\end{aligned} \tag{B4}$$

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