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Quasi-analytic Multidimensional Signals

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Abstract: In a recent paper [1], the authors presented the unified theory of n-D complex and hypercomplex analytic signals with single-orthant spectra. This paper describes a specific form of these signals called quasi-analytic. A quasi-analytic signal is a product of a low-pass (base-band) real (in general non-separable) signal and a n-D complex or hypercomplex carrier (modulation). By a suitable choice of a carrier frequency, the spectrum is shifted into a single orthant of the Fourier frequency space with a negligible leakage in other orthants. A measure of this leakage is defined. Properties of quasi-analytic signals are presented. In implementations, Hilbert filters are replaced by modulators. Problems of polar representation of quasi-analytic signals and of its lower rank representation are presented.

1 Introduction

Multidimensional analytic signals with single-orthant spectra have been introduced in [2] and described in [3] and [4]. A unified theory of complex and hypercomplex analytic signals with single-orthant spectra has recently been presented in [1]. Here, we describe specific implementations of analytic signals in the form of quasi-analytic signals. Let us remind the basic definition of analytic signals with single-orthant spectra. Consider a real n-D signal $g(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and its n-D complex or hypercomplex Fourier transform $g(\boldsymbol{x}) \overset{\mathcal{F}}{\Leftrightarrow} G(\boldsymbol{f})$, $\boldsymbol{f} = (f_1, f_2, ..., f_n)$. The multiplication of the spectrum $G(\boldsymbol{f})$ by a n-D unit-step $\mathbf{1}(\boldsymbol{f})$ yields a single-orthant spectrum in the first orthant of the frequency space $\mathbb{R}^n$, i.e., in $\mathbb{R}^+$.

The n-D complex or hypercomplex analytic signal is defined by the complex or hypercomplex inverse Fourier transform :

$$\psi_1(\boldsymbol{x}) = \text{Inverse Fourier Transform of } \{\mathbf{1}(\boldsymbol{f})\, G(\boldsymbol{f})\}. \tag{1}$$

The quasi-analytic signal is defined by the formula

$$\psi_1(\boldsymbol{x}) = g(\boldsymbol{x}) e^{e_1 2\pi f_{10} x_1} e^{e_2 2\pi f_{20} x_2} \cdots e^{e_n 2\pi f_{n0} x_n}, \tag{2}$$

i.e., the real signal $g(\boldsymbol{x})$ is multiplied by a multidimensional carrier. The spectrum is shifted into the first orthant having the form

$$G_s(\boldsymbol{f}) = G(f_1 - f_{10}, f_2 - f_{20}, ..., f_n - f_{n0}). \tag{3}$$

The all carrier frequencies f_{i0} should be positive constants and enable the effective shift of the spectrum in the first orthant. The shifted spectrum should have a negligible leakage of its support outside the first orthant. A measure of the leakage is represented by the coefficient ε defined by the equation

$$\varepsilon = 1 - \frac{\int\limits_{\mathbb{R}^+} |G_s|^2 \, d\,\boldsymbol{f}}{\int\limits_{\mathbb{R}_n} |G|^2 \, d\,\boldsymbol{f}} = 1 - \frac{\text{Energy of the signal in the first orthant}}{\text{Total energy of the signal}}. \tag{4}$$

For quasi-analytic signals $\varepsilon \ll 0$. If the support of the spectrum $G(\boldsymbol{f})$ is finite, it is possible to get $\varepsilon = 0$. Note that by an appropriate change of a sign of carrier frequencies f_{i0} we can define quasi-analytic signals with spectra in other orthants of the frequency space. The signal (2) is called complex if all basis vectors are the same (usual notation j) with $j^2 = -1$. The form of hypercomplex signals is defined by te choice of the algebra of basis vectors [1].

2 2-D complex quasi-analytic signals

Let us illustrate the notion of a 2-D complex quasi-analytic signal with two examples of low-pass real signals: the 2-D Gaussian one and a rotated cuboid. The 2-D Gaussian signal (normalized form) has the form

$$g(x_2, x_1) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2 + 2\rho x_1 x_2 / (\sigma_1\sigma_1)}{2(1-\rho^2)} \right]. \tag{5}$$

It is a nonseparable function of variables (x_1, x_2). If $\rho = 0$, it is a separable function $g(x_1, x_2) = g_1(x_1) g_2(x_2)$. Generally, a 2-D signal is a union of four terms with different even-odd parity. However, low-pass signals are unions of only two terms: $g(x_1, x_2) = g_{ee}(x_1, x_2) + g_{oo}(x_1, x_2)$, where "*ee*" denotes an even-even term and "*oo*" an odd-odd term w.r.t. variables (x_1, x_2). If $\rho = 0$, the odd-odd term vanishes. The Fourier spectrum of (5) is ($\omega_i = 2\pi f_i$) :

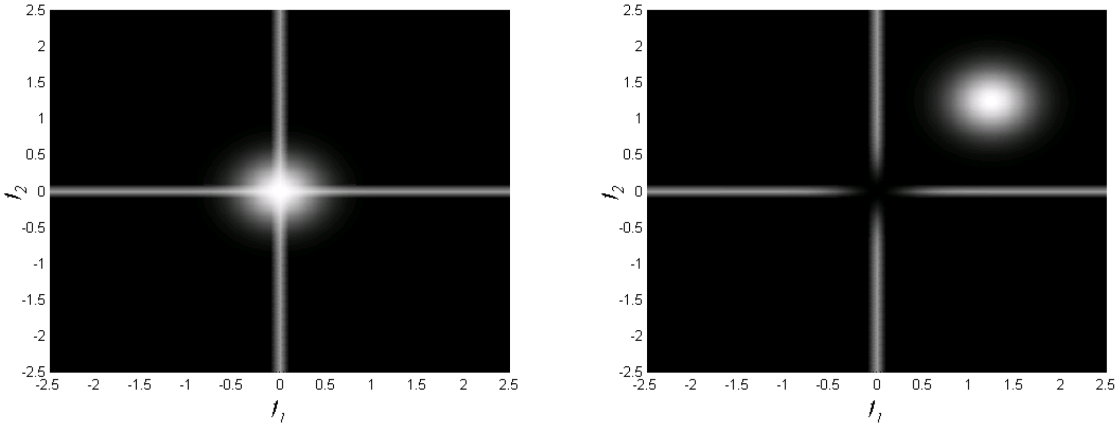
$$G(f_1, f_2) = \exp\left[-0.5\left(\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\rho\omega_1\omega_2\sigma_1\sigma_2 \right) \right] = G_{ee}(f_1, f_2) + G_{oo}(f_1, f_2) \tag{6}$$

where

$$G_{ee}(f_1, f_2) = \exp\left[-0.5\left(\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 \right) \right] \cosh(\omega_1\omega_2\sigma_1\sigma_2\rho), \tag{7}$$

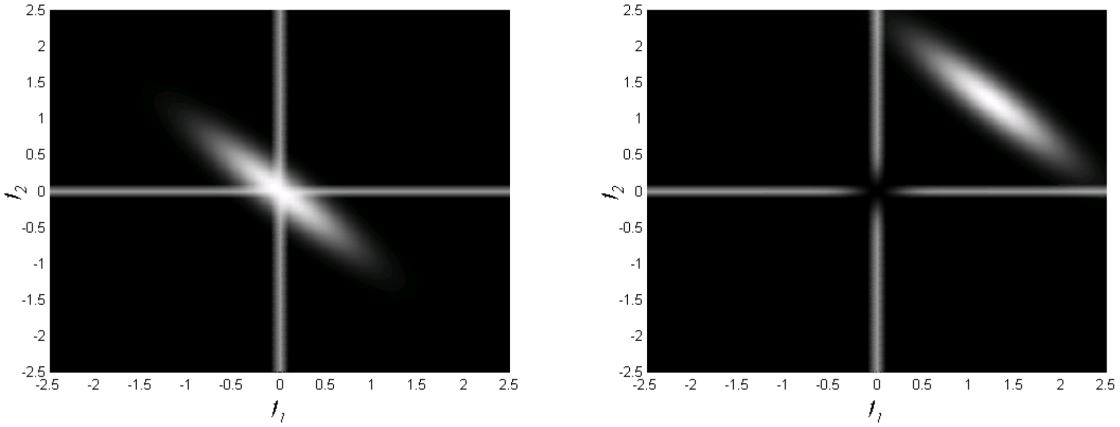
$$G_{oo}(f_1, f_2) = \exp\left[-0.5\left(\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 \right) \right] \sinh(\omega_1\omega_2\sigma_1\sigma_2\rho). \tag{8}$$

Let us present some specific examples of Gaussian spectra. Fig. 1a shows the low-pass spectrum for $\sigma_1 = \sigma_2 = 0.5$ and $\rho = 0$. The spectrum of the quasi-analytic signal shifted by modulation into the first quadrant ($f_1 > 0$, $f_2 > 0$).is shown in Fig. 1b.



(a) (b)

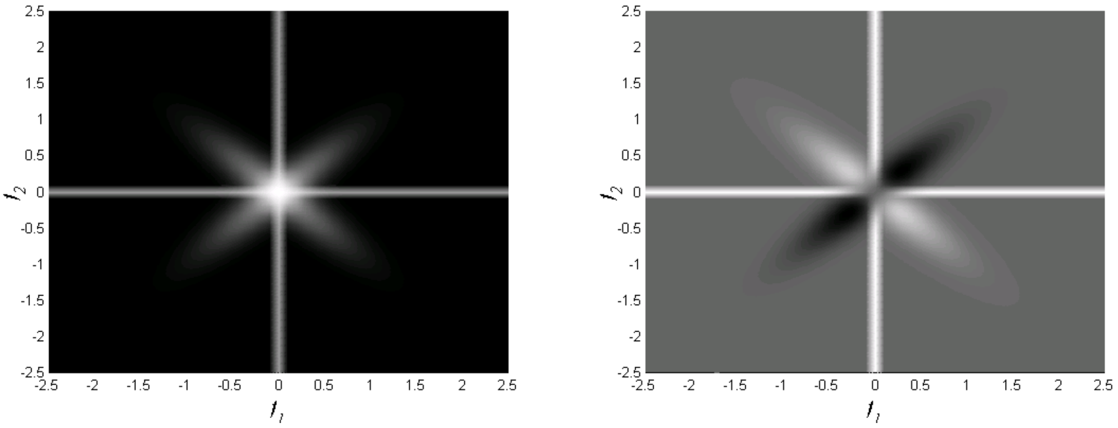
Fig. 1 (a) The Gaussian signal $\sigma_1 = \sigma_2 = 0.5$, $\rho = 0$, $G(f_1, f_2) = G_{ee}$, (b) The shifted spectrum of (a), $G(f_1 - 1.25, f_2 - 1.25)$, no leakage, $\varepsilon \to 0.00000$



(a) (b)

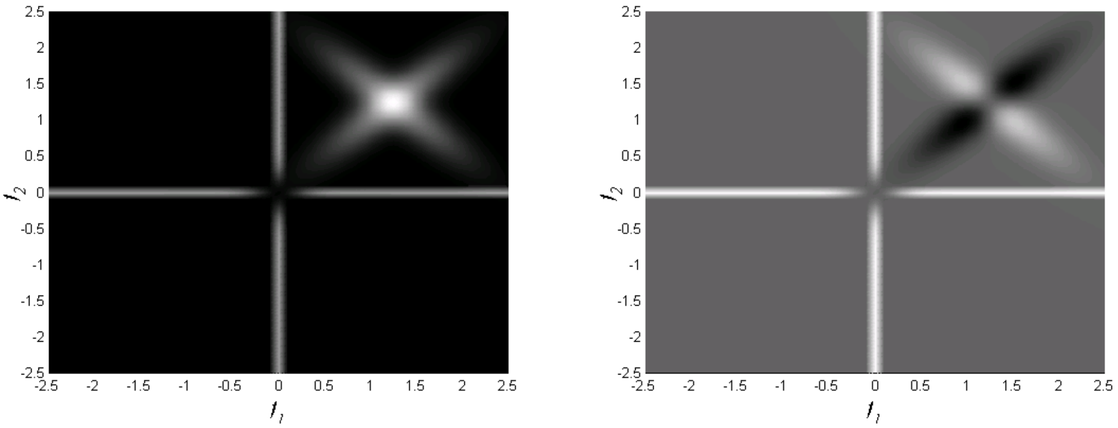
Fig. 2 (a) The Gaussian signal $\sigma_1 = \sigma_2 = 0.7$, $\rho = 0.9$, $G(f_1, f_2) = G_{ee} + G_{oo}$, (b) The shifted spectrum of (a), $G(f_1 - 1.25, f_2 - 1.25)$, negligible leakage, $\varepsilon = 0.00054$

Fig. 2a shows the spectrum with $\sigma_1 = \sigma_2 = 0.7$, $\rho = 0.9$ (nonseparable case). The even-even and odd-odd terms are displayed in Figs 3a,b. Fig. 4 a,b shows the same spectra shifted into the first quadrant.



(a) (b)

Fig. 3 (a) The even-even part of the spectrum of Fig.2a, (b) The odd-odd part of the spectrum of Fig.2a



(a) (b)

Fig. 4 (a) The even-even part of the spectrum of Fig.2a shifted into the 1st quadrant, (b) The odd-odd part of the spectrum of Fig.2a shifted into the 1st quadrant

Note that energies of low-pass signals in the non-separable case (Fig.2a) are different in the 1st quadrant ($f_1 > 0$, $f_2 > 0$) and in the 3rd quadrant ($f_1 > 0$, $f_2 < 0$). In consequence, the inverse Fourier transform of these two single-quadrant spectra define two different analytic signals:

$$\psi_1(x_1,x_2) = \mathcal{F}^{-1}\{\mathbf{1}(f_1,f_2)G(f_1,f_2)\} = g(x_1,x_2) - v(x_1,x_2) + e_1\left[v_1(x_1,x_2) + v_2(x_1,x_2)\right] \quad (9)$$

$$\psi_3(x_1,x_2) = \mathcal{F}^{-1}\{\mathbf{1}(f_1,-f_2)G(f_1,f_2)\} = g(x_1,x_2) + v(x_1,x_2) + e_1\left[v_1(x_1,x_2) - v_2(x_1,x_2)\right] \quad (10)$$

Notations: $g(x_1,x_2)$ is a real signal, $v(x_1,x_2)$ - the total (w.r.t. both variables) Hilbert transform of u, $v_1(x_1,x_2)$- the partial Hilbert transform w.r.t. x_1 and $v_2(x_1,x_2)$ the partial Hilbert transform w.r.t. x_2.

The 2-D quasi-analytic signal has the form

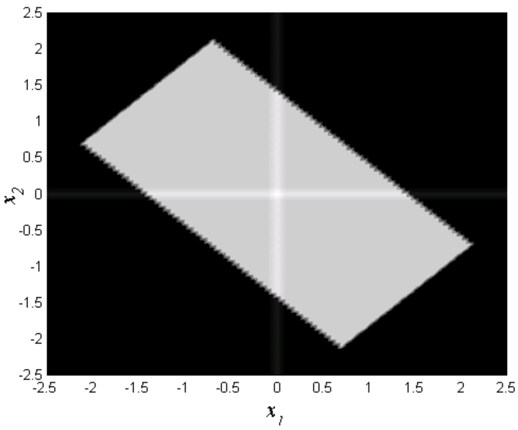
$$\psi_1\left(x_1,x_2\right)\approx g\left(x_1,x_2\right)e^{e_1 2\pi f_{10}}e^{e_1 2\pi f_{20}}\,. \tag{11}$$

The developed form of (11) is

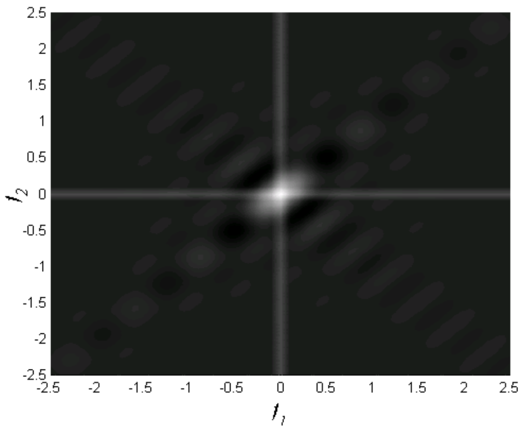
$$\psi_1\left(x_1,x_2=\right)g\left(x_1x_2\right)\left[c_1c_2-s_1s_2+e_1\left(s_1c_2+s_2c_1\right)\right]=\mathrm{Re}+e_1\,\mathrm{Im} \tag{12}$$

where $c_i=\cos\left(\alpha_i\right)$, $s_i=\sin\left(\alpha_i\right)$, $\alpha_i=2\pi f_{i0}x_i$.

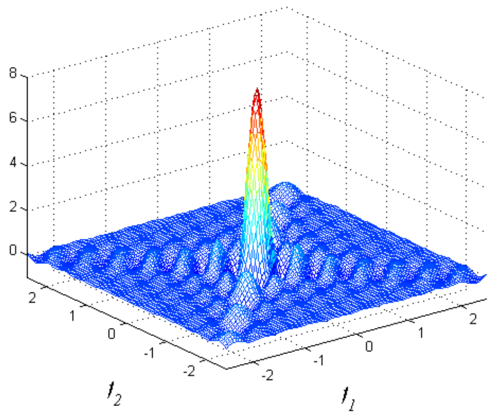
Let us present the next example using a rotated cuboid (see Appendix A). The support of this signal is displayed in Fig. 5a and its spectrum in Figs 5b and c. The spectrum of the quasi-analytic signal shifted into the 1st quadrant is shown in Figs 6a and b. Due to the sharp edges of the cuboid, the oscillatory spectrum has a finite leakage into other quadrants with ε = 0.013.



(a)

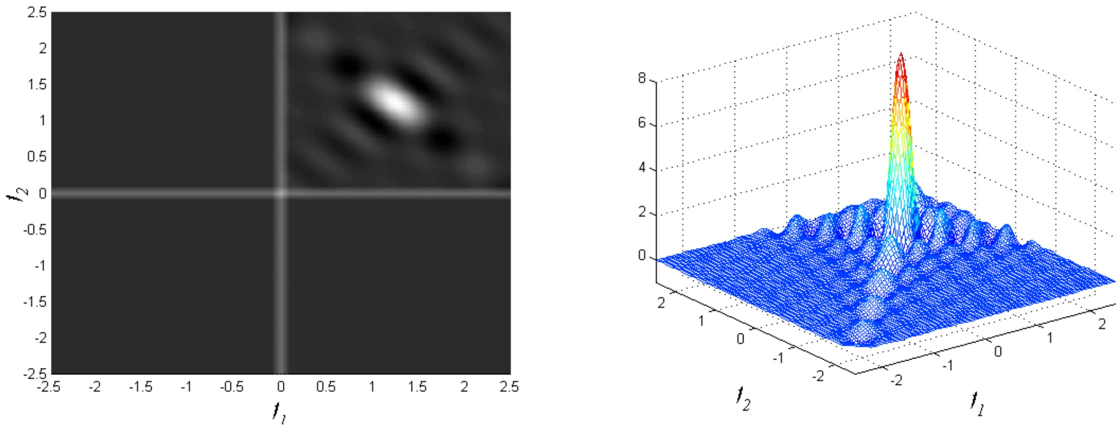


(b)



(c)

Fig. 5 (a) The support of a rotated cuboid, (b) The spectrum of the cuboid of (a), (c) The 3-D view to the spectrum of Fig.5b



(a) (b)

Fig. 6 (a) The spectrum of the quasi-analytic signal of the rotated cuboid, (b) The 3-D view to the shifted spectrum of the rotated cuboid

3 Polar representation of 2-D complex quasi-analytic signals

The polar forms of complex analytic signals (9) and (10) define two different amplitudes and two different phase functions. However, if $g(x_1, x_2)$ is a separable function, we have a single amplitude and two phase functions, the first one is $\phi_1 = \alpha_1(x_1) + \alpha_2(x_2)$ and the second $\phi_2 = \alpha_1(x_1) - \alpha_2(x_2)$. Let us remind that polar forms of analytic signals are uniquely defined. Therefore, by defining the polar forms of quasi-analytic signals we should recall that they are analytic only approximately, if $\varepsilon << 1$. Since the quasi-analytic signals are separable functions, we have a single amplitude and two phase functions.

The polar representation of (11) is

$$\psi_1(x_1, x_2) = A(x_1, x_2) e^{e_1 \phi(x_1, x_2)} = |g(x_1, x_2)| e^{e_1 \phi(x_1, x_2)}. \tag{13}$$

The local amplitude of (13) is

$$A(x_1, x_2) = \sqrt{\psi_1 \psi_1^*} = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = |g(x_1, x_2)| \tag{14}$$

and the phase

$$\tan(\phi) = \mathrm{tg}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \mathrm{tg}(\alpha_1 + \alpha_2), \tag{15}$$

i.e., as expected, $\phi = \alpha_1 + \alpha_2$. Note that (13) differs from (11) since $g(x_1, x_2)$ is replaced by the absolute value. For unipolar positive baseband functions they are equal. The derivation of (15) is presented only for formal reasons, since the phase functions of (11) and (13) are the same. The Hilbert transforms of $u(x_1, x_2) = g(x_1, x_2) c_1 c_2$ have the form

$v(x_1,x_2) \approx g(x_1,x_2)s_1 s_2$, $v_1(x_1,x_2) \approx g(x_1,x_2)s_1 c_2$ and $v_2(x_1,x_2) \approx g(x_1,x_2)c_1 s_2$. Notations: $c_1 = \cos(2\pi f_{10} x_1)$, $c_2 = \cos(2\pi f_{20} x_2)$, $s_1 = \sin(2\pi f_{10} x_1)$ and $s_2 = \sin(2\pi f_{20} x_2)$. We observe that the transforms are obtained using $H\left[\cos(\alpha_i)\right] = \sin(\alpha_i)$.

4. 2-D quaternionic quasi-analytic signals

Differently to the real spectrum of the real signal $g(x_1, x_2)$, e.g. given by (6), the corresponding quaternionic spectrum is a complex function:

$$G_q(f_1, f_2) = G_{ee}(f_1, f_2) + e_3 G_{oo}(f_1, f_3). \tag{16}$$

In consequence, if both terms of the spectra exist, the even-even and odd-odd terms of (16) should be shifted into the first quadrant separately. The quaternionic quasi-analytic signal has the form

$$\psi_{q1}(x_1,x_2) = g(x_1,x_2)e^{e_1\alpha_1}e^{e_2\alpha_2}, \tag{17}$$

i.e., in comparison to (11) in the second exponent e_1 is replaced by e_2. The developed form of (17) is

$$\psi_q(x_1,x_2) = g(x_1,x_2)\left(\left(c_1c_2 + e_1 s_1 c_2 + e_2 c_1 s_2 + e_3 s_1 s_2\right)\right). \tag{18}$$

5. Polar representation of 2-D quaternionic quasi-analytic signals

The polar representation of the quaternion derived in [5] has the form

$$\psi_q(x_1,x_2) = Ae^{e_1\phi_1}e^{e_3\phi_3}e^{e_2\phi_2}. \tag{19}$$

The amplitude is

$$A(x_1,x_2) = \sqrt{\psi\psi^*} = \left|g(x_1,x_2)\right| \tag{20}$$

i.e., equals (14), and the three phase functions (Euler angles) are

$$\tan(2\phi_1) = \frac{2(uv_1 + vv_2)}{u^2 - v_1^2 + v_2^2 - v^2} = 2\frac{c_1c_2s_1c_2 + s_1s_2c_1s_2}{c_1^2c_2^2 - s_1^2c_2^2 + c_1^2s_2^2 - s_1^2s_2^2} = \frac{2\tan(\alpha_1)}{1-\tan^2(\alpha_1)} = \tan(2\alpha_1), \tag{21}$$

$$\tan(2\phi_2) = \frac{2(uv_2 + vv_1)}{u^2 - v_1^2 + v_2^2 - v^2} = 2\frac{c_1c_2c_1s_2 + s_1c_2s_1s_2}{c_1^2c_2^2 + s_1^2c_2^2 - c_1^2s_2^2 - s_1^2s_2^2} = \frac{2\tan(\alpha_2)}{1-\tan^2(\alpha_2)} = \tan(2\alpha_2), \tag{22}$$

$$\sin(\phi_3) = \frac{uv - v_1v_2}{A^2} = \frac{c_1c_2s_1s_2 - s_1c_2c_1s_2}{A^2} = 0. \tag{23}$$

Therefore, $\phi_1 = \alpha_1$; $\phi_2 = \alpha_2$ and $\phi_3 = 0$. We have

$$\psi_q(x_1, x_2) \approx |g(x_1, x_2)| e^{e_1 2\pi f_{10} x_1} e^{e_2 2\pi f_{20} x_2} . \tag{24}$$

Similarly to the complex case, we have only two Euler angles. This could be derived directly by comparison of (17) and (24).

6 Lower rank 2-D signals

Lower rank 2-D signals have been defined in [1] in the form of a union of two signals with single-quadrant spectra in the first and third quadrant, i.e., with the spectrum defned in the half plane $f_1 > 0$. The rank-1 signals are

$$\psi_{HS}(x_1, x_2) = \frac{\psi_1 + \psi_3}{2} = \frac{\psi_{1q} + \psi_{3q}}{2} = u(x_1, x_2) + e_1 v_1(x_1, x_2) = A e^{e_1 \phi} , \tag{25}$$

i.e. the rank-1 2-D signals have the same form for complex and hypercomplex (quaternionic) signals. Their amplitude is $A = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = |g(x_1, x_2)| |\cos(\alpha_2)|$ and the phase is $\phi(x_1, x_2) = \alpha_1$.

7 3-D complex quasi-analytic signals.

Similarly to the 2-D case, a 3-D quasi-analytic signal is defined by the inverse Fourier transform of a low-pass spectrum of a real signal shifted into a single octant. The complex quasi-analytic signal with single octant spectrum in the first octant has the form

$$\psi_1(x_1, x_2, x_3) = g(x_1, x_2, x_3) e^{e_1 \alpha_1} e^{e_1 \alpha_2} e^{e_1 \alpha_3} , \tag{26}$$

$\alpha_i = 2\pi f_{i0} x_i$ and f_{i0} are three shift frequencies of the carrier. The developed form of (26) is

$$\psi_1(x_1, x_2, x_3) = g(x_1, x_2, x_3)\left(c_1 c_2 c_3 - s_1 s_2 c_3 - s_1 c_2 s_3 - c_1 s_2 s_3\right) + e_1\left(s_1 c_2 c_3 + c_1 s_2 c_3 + c_1 c_2 s_3 - s_1 s_2 s_3\right) \tag{27}$$

where $c_i = \cos(\alpha_i)$ and $s_i = \sin(\alpha_i)$. Let us remind that the Hilbert transforms are calculated using $s_i = H(c_i)$. Consider the example with a 3-D nonseparable Gaussian signal (corresponding formulae are given in Appendix B). This signal is defined by three variances σ_1 , σ_2 , σ_3 and three correlation coefficients ρ_{12}, ρ_{13}, ρ_{23}. A specific example of the Gaussian low-pass spectrum is shown in Fig. 7a. Fig. 7b shows the cross-section of the spectrum of Fig.5a shifted into the first octant (Remark: we present only a chosen cross-section of a 3-D spectrum).

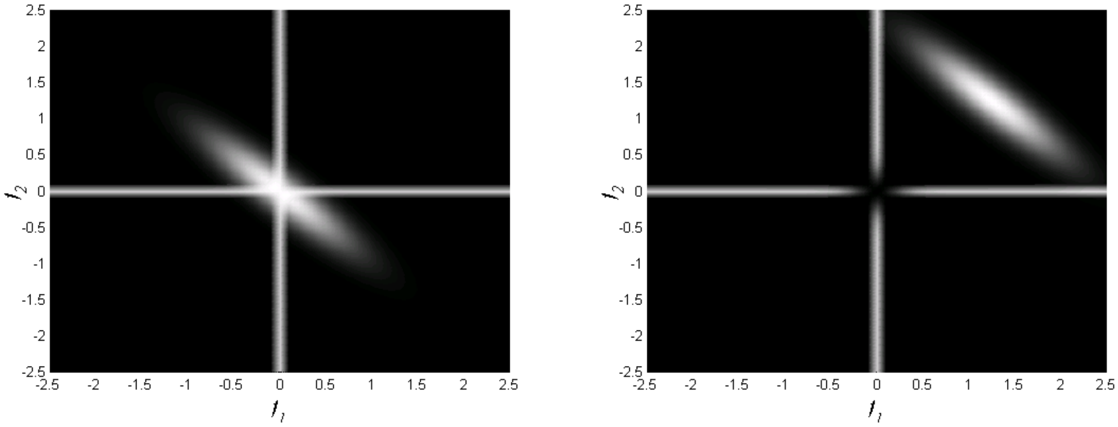


Fig. 7 (a) The cross-section $G(f_1, f_2, f_3{=}0)$ of the spectrum of a Gaussian 3-D signal $\sigma_1 = \sigma_2 = \sigma_3 = 0.7$, $\rho_{12} = \rho_{13} = \rho_{23} = 0.9$. No shift (b) The shifted spectrum of (a), $G(f_1\text{-}1.25, f_2\text{-}1.25, f_3{=}0)$. Leakage $\varepsilon = 0.0000$.

Let us recall that a 3-D signal may be represented as a sum of 8 terms with different parity (even/odd). However, low-pass real signals are unions of only four terms [1]:

$$g(x_1, x_2, x_3) = g_{\text{eee}} + g_{\text{eoo}} + g_{\text{oeo}} + g_{\text{ooe}}; \qquad G(f_1, f_2, f_3) = G_{\text{eee}} - G_{\text{eoo}} - G_{\text{oeo}} - G_{\text{ooe}}. \quad (28)$$

First, let us compare the 3-D analytic signals with quasi-analytic signals. In the half-space $f_1 > 0$ we have four octants labbelled 1, 3, 5 and 7. Their energies may differ. In consequence, a real signal g is represented by four different analytic signals with single-octant spectra denoted ψ_1, ψ_3, ψ_5 and ψ_7 of different energies. For example, the signal ψ_1 has the form

$$\psi_1(x_1, x_2, x_3) = g - v_{12} - v_{13} - v_{23} + e_1(v_1 + v_2 + v_3 - v). \quad (29)$$

Notations are similar as in the 2-D case: v_i denotes partial Hilbert transforms of g w.r.t. a single variable and v_{ij} w.r.t. two variables. The signals ψ_3, ψ_5 and ψ_7 differ only by signs (see [2], [3], [4]). However, the four quasi-analytic signals defined (26) differ for other octants only by the signs of the angles in the exponents. All have the same amplitude and the same energies.

8 Polar representation of 3-D complex quasi-analytic signals

The polar form of four complex analytic signal defines four amplitudes and four phase functions. For separable signals all four amplitudes are equal and the four phase functions are

$$\phi_1 = \alpha_1(x_1) + \alpha_2(x_2) + \alpha_3(x_3), \qquad \phi_3 = \alpha_1(x_1) - \alpha_2(x_2) + \alpha_3(x_3), \quad (30)$$

$$\phi_5 = \alpha_1(x_1) + \alpha_2(x_2) - \alpha_3(x_3), \qquad \phi_7 = \alpha_1(x_1) - \alpha_2(x_2) - \alpha_3(x_3) \quad (31)$$

and the amplitude

$$A(x_1,x_2,x_3)=\sqrt{\psi_1\psi_1^*}=\sqrt{\mathrm{Re}^2+\mathrm{Im}^2}=\left|g(x_1,x_2,x_3)\right| \tag{32}$$

The phase function of ϕ_1 is

$$\tan(\phi_1)=\mathrm{tg}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)=\frac{s_1c_2c_3+c_1s_2c_3+c_1c_2s_3-s_1s_2s_3}{c_1c_2c_3-s_1s_2c_3-s_1c_2s_3-c_1s_2s_3}=\mathrm{tg}(\alpha_1+\alpha_2+\alpha_3). \tag{33}$$

9 3-D hypercomplex quasi-analytic signals

9.1 Cayley-Dickson algebra

The 3-D quasi-analytic hypercomplex signal defined by the Cayley-Dickson algebra of unit vectors (see Appendix C) is

$$\psi_{CD}(x_1,x_2,x_3)=g(x_1,x_2,x_3)e^{e_1\alpha_1}e^{e_2\alpha_2}e^{e_4\alpha_3} \tag{34}$$

where again $\alpha_i= 2\pi f_{i0}x_i$. Note the order e_1, e_2, e_4. The developed form of (34) is

$$\begin{aligned}&\psi_{CD}(x_1,x_2,x_3)\\&=g(x_1,x_2,x_3)(c_1c_2c_3+e_1s_1c_2c_3+e_2c_1s_2c_3+e_3s_1s_2c_3+e_4c_1c_2s_3+e_5s_1c_2s_3+e_6c_1s_2s_3\pm e_6s_1s_2s_3)\end{aligned} \tag{35}$$

Remark: The sign of e_7 depends on the order of multiplication.

9.2 Clifford algebra

The 3-D quasi-analytic signal defined by the Clifford algebra of unit vectors (see Appendix D) is

$$\psi_{CD}(x_1,x_2,x_3)=g(x_1,x_2,x_3)e^{e_1\alpha_1}e^{e_2\alpha_2}e^{e_3\alpha_3} \tag{36}$$

In comparison to (34) the order is e_1, e_2, e_3 The developed form of (31) is

$$\begin{aligned}\psi_{Cl}(x_1,x_2,x_3)=g(x_1,x_2,x_3)\big(&c_1c_2c_3+e_1s_1c_2c_3+e_2c_1s_2c_3+(e_1e_2)s_1s_2c_3+e_3c_1c_2s_3\\&+(e_1e_3)s_1c_2s_3+(e_2e_3)c_1s_2s_3+\omega s_1s_2s_3\big)\end{aligned} \tag{37}$$

Assuming that the amplitude should be an unipolar positive function, the polar form of (37) is undefined, since $\psi_{Cl}\psi_{Cl}^*=g^2(x_1,x_2,x_3)\left(1-2s_1^2s_2^2s_3^2\right)$ is a bipolar function. This is caused by the fact that in the Clifford algebra (see Appndix D) ω^2 = +1 and not –1.

10 Polar representation of 3-D hypercomplex quasi-analytic signals

The problem of the polar representation of 3-D hypercomplex analytic signals has been discussed in [1]. In principle, the 3-D hypercomplex signal with single-octant spectrum is represented by a single amplitude and seven phase functions. However, if the signal is a separable function, we have only three phase functions. Since quasi-analyic signals are separable functions, the signal (34) is represented by three phase functions. Its amplitude is

$$A(x_1, x_2, x_3) = \sqrt{\psi_{CD}\psi_{CD}^{*}} = \left|g\left(x_1, x_2, x_3\right)\right| \tag{38}$$

and the three phase angles, as shown in [1], are α_1, α_2, α_3. Therefore,

$$\psi_{CD}\left(x_1, x_2, x_3\right) = \left|g\left(x_1, x_2, x_3\right)\right| e^{e_1\alpha_1} e^{e_2\alpha_2} e^{e_4\alpha_3} . \tag{39}$$

Conclusion: From the point of view of the polar representation, the 2-D and 3-D quasi-analytic signals of non-separable low-pass signals have the analogous polar representation as separable functions: We have a single amplitude and two (in 2-D) or three (in 3-D) phase functions.

10.1 Lower rank 3-D signals

The above described 3-D signals have the rank 3. Let us derive the formulae for the signals of rank 2.

a) ***Complex case***

The complex signal (26) has the spectral support in the first octant. Let us write signals with spectral supports in the octants No. 3, 5 and 7: $\psi_3 = ge^{e_1\alpha_1}e^{-e_1\alpha_2}e^{e_1\alpha_3}$, $\psi_3 = ge^{e_1\alpha_1}e^{e_1\alpha_2}e^{-e_1\alpha_3}$ and $\psi_3 = ge^{e_1\alpha_1}e^{-e_1\alpha_2}e^{-e_1\alpha_3}$. The signals of rank 2 are

$$\psi_{1+5}\left(x_1, x_2, x_3\right) = \frac{\psi_1 + \psi_5}{2} = ge^{e_1\alpha_1}e^{e_1\alpha_2}\cos\left(\alpha_3\right) = A_{1+5}{}^{e_1\phi_{1+5}} \tag{40}$$

$$\psi_{3+7}\left(x_1, x_2, x_3\right) = \frac{\psi_3 + \psi_7}{2} = ge^{e_1\alpha_1}e^{-e_1\alpha_2}\cos\left(\alpha_3\right) = A_{3+7}{}^{e_1\phi_{3+7}} \tag{41}$$

The amplitudes are the same: $A_{1+5} = A_{3+7} = |g|\left|\cos\left(\alpha_3\right)\right|$ and the phase functions are $\phi_{1+5} = \alpha_1 + \alpha_2$, $\phi_{3+7} = \alpha_1 - \alpha_2$. The rank-1 signal with the support of its spectrum in the half space $f_1 > 0$ is

$$\psi_{HS}\left(x_1, x_2, x_3\right) = \frac{\psi_{1+5} + \psi_{3+7}}{2} = g\cos\left(\alpha_2\right)\cos\left(\alpha_3\right)e^{e_1\phi_1} = A_{HS}e^{e_1\phi_{HS}} . \tag{42}$$

Its amplitude $A_{HS} = |g|\left|\cos\left(\alpha_2\right)\right|\left|\cos\left(\alpha_3\right)\right|$ and the phase function is $\phi_{HS} = \alpha_1$.

b) ***Octonion case (Cayley-Dickson algebra)***

The hypercomplex signal (39) has the spectral support in the first octant. Let us write signals with spectral supports in the octants No. 3, 5 and 7: $\psi_3 = ge^{e_1\alpha_1}e^{-e_2\alpha_2}e^{e_4\alpha_3}$, $\psi_3 = ge^{e_1\alpha_1}e^{e_2\alpha_2}e^{-e_4\alpha_3}$ and $\psi_3 = ge^{e_1\alpha_1}e^{-e_2\alpha_2}e^{-e_4\alpha_3}$. The signals of rank 2 are

$$\psi_{1+5}\left(x_1,x_2,x_3\right)=\frac{\psi_1+\psi_5}{2}=ge^{e_1\alpha_1}e^{e_2\alpha_2}\cos\left(\alpha_3\right)=A_{1+5}e^{e_1\phi_{1+5}^{(1)}}e^{e_2\phi_{1+5}^{(2)}}, \tag{43}$$

$$\psi_{3+7}\left(x_1,x_2,x_3\right)=\frac{\psi_3+\psi_7}{2}=ge^{e_1\alpha_1}e^{-e_2\alpha_2}\cos\left(\alpha_3\right)=A_{3+7}e^{e_1\phi_{3+7}^{(1)}}e^{-e_2\phi_{3+7}^{(2)}}. \tag{44}$$

The amplitudes are the same as in (41) and (43) an the phase functions expressed by Euler angles given by (21) and (22) are $\phi_{1+5}^{(1)}\left(x_1,x_2.x_3\right)=\phi_{3+7}^{(1)}=\alpha_1$, $\phi_{1+5}^{(2)}\left(x_1,x_2.x_3\right)=-\phi_{3+7}^{(2)}=\alpha_2$. Of course, the phase angles are defined directly by the comparison of exponents in (43) and (44). However, using the Euler angles (21)-(23) we can show that the same formulae apply for quaternions with 3-D terms. The rank-1 signals are again the same for complex and octonic signals. We have

$$\psi_{HS}\left(x_1,x_2,x_3\right)=\frac{\psi_{1+5}+\psi_{3+7}}{2}=g\cos\left(\alpha_2\right)\cos\left(\alpha_3\right)e^{e_1\alpha_1}=A_{HS}e^{e_1\phi_{HS}}. \tag{45}$$

The amplitude is $A_{HS}=\left|g\right|\left|\cos\left(\alpha_2\right)\right|\left|\cos\left(\alpha_3\right)\right|$ and the phase $\phi_{HS}=\alpha_1$.

11 Conclusions

Quasi-analytic signals with single-orthant spectra are defined by multplication of a low-pass (baseband) nonseparable n-D signal $g(x_1, x_2,\ldots,x_n)$ by a muldimimensional carrier. This operation should shift the low-pass spectrum of g into a single-orthant. The leakage of the energy of the modulated signal in other orthants of the frequency space should be negligible. The modulating carrier can be complex or hypercomplex. The paper presents details of this method for 2-D and 3-D signals.

References:

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Appendix A The spectrum of a rotated cuboid

The spectrum of the cuboid is well known. For convenience, let us recall the definition of a cuboid. The symmetric cuboid (nonrotated) is defined as a product of two rectangles

$$g(x_2,x_1)=\Pi_a(x_1)\Pi_b(x_2),\quad \Pi_a(x_1)=\begin{matrix} 1 & if & |x_1|<a \\ 0.5 & if & |x_1|=a \\ 0 & if & |x_1|>a \end{matrix}\ ,\quad \Pi_a(x_1)=\begin{matrix} 1 & if & |x_1|<a \\ 0.5 & if & |x_1|=a \\ 0 & if & |x_1|>a \end{matrix} \tag{A1}$$

The Fourier spectrum of (A1) is

$$G(f_2,f_1)=2a\frac{\sin(2\pi f_1 a)}{2\pi f_1 a}2b\frac{\sin(2\pi f_2 b)}{2\pi f_2 b} \tag{A2}$$

Of course g and G are separable 2-D functions. The rotated cuboid is defined in a coordinate system rotated by the angle γ as

$$x_1^{'}=x_1\cos(\gamma)+x_2\sin(\gamma)\quad ;\quad x_2^{'}=x_2\cos(\gamma)-x_1\sin(\gamma) \tag{A3}$$

and is nonseparable. As well, the spectrum of the rotated cuboid is defined by rotation of the frequency domain coordinate system

$$f_1^{'}=f_1\cos(\gamma)+f_2\sin(\gamma)\quad ;\quad f_2^{'}=f_2\cos(\gamma)-f_1\sin(\gamma) \tag{A4}$$

and is also a nonseparable function.

Appendix B The 3-D Gaussian signal

The 3-D Gaussian signal is defined by

$$u(x_1,x_2,x_3)=(2\pi)^{-3/2}|M|^{-1/2}\exp\left\{\frac{-1}{2|M|}\sum_{i,j=1}^{3}|M_{ij}|x_i x_j\right\} \tag{B1}$$

where

$|M|=\sigma_1^2\sigma_2^2\sigma_3^2\left(1+\rho_{12}\rho_{23}\rho_{13}+\rho_{13}\rho_{12}\rho_{23}-\rho_{12}^2-\rho_{23}^2-\rho_{13}^2\right)$, $|M_{11}|=\left(1-\rho_{23}^2\right)\sigma_2^2\sigma_3^2$,

$|M_{22}|=\left(1-\rho_{13}^2\right)\sigma_1^2\sigma_3^2$, $|M_{33}|=\left(1-\rho_{12}^2\right)\sigma_1^2\sigma_2^2$, $|M_{12}|=|M_{21}|=\sigma_1\sigma_2\sigma_3^2\left(\rho_{23}\rho_{13}-\rho_{12}\right)$,

$|M_{23}|=|M_{32}|=\sigma_1^2\sigma_2\sigma_3\left(\rho_{12}\rho_{13}-\rho_{23}\right)$ and $|M_{13}|=|M_{31}|=\sigma_1\sigma_2^2\sigma_3\left(\rho_{12}\rho_{23}-\rho_{13}\right)$.

The parameters σ_i^2, i = 1, 2, 3 are called variances and ρ_{ij}, i, j = 1, 2, 3, $i \neq j$ are cross-correlation factors. If all $\rho_{ij} = 0$, we have a 3-D separable Gaussian signal. The Fourier spectrum of (B1) is

$$U(\omega_1, \omega_2, \omega_3) = \exp\left[-\tfrac{1}{2}\left(\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + \omega_3^2\sigma_3^2\right)\right]\exp\left[-\left(\omega_1\omega_2\rho_{12}\sigma_1\sigma_2 + \omega_1\omega_3\rho_{13}\sigma_1\sigma_3 + \omega_2\omega_3\rho_{23}\sigma_2\sigma_3\right)\right]. \quad \text{(B2)}$$

Appendix C The Cayley-Dickson algebra

The Cayley-Dickson multiplication rules of unit vectors are presented in Table 1. Details concerning the Cayley-Dickson algebra are presented in [1] or in many other sources. Note that the part of the table for e_1, e_2 and e_3 presents multiplication rules of quaternions.

TABLE 1 MULTIPLICATION RULES IN THE ALGEBRA OF OCTONIONS

$\times$	***1***	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_4}$	$\boldsymbol{e_5}$	$\boldsymbol{e_6}$	$\boldsymbol{e_7}$
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e_1}$	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
$\boldsymbol{e_4}$	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
$\boldsymbol{e_5}$	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
$\boldsymbol{e_6}$	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
$\boldsymbol{e_7}$	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Appendix D The Clifford algebra $Cl_{0,3}(\mathbb{R})$

The rules of multiplication of unit vectors of the Clifford algebra $Cl_{0,3}(\mathbb{R})$ are given in Table 2. Details can be found in [1] or in other sources.

TABLE 2 MULTIPLICATION RULES IN $Cl_{0,3}(\mathbb{R})$

×	**1**	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_1e_2}$	$\boldsymbol{e_1e_3}$	$\boldsymbol{e_2e_3}$	$\boldsymbol{\omega}$
1	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	ω
$\boldsymbol{e_1}$	e_1	-1	e_1e_2	e_1e_3	$-e_2$	$-e_3$	ω	$-e_2e_3$
$\boldsymbol{e_2}$	e_2	$-e_1e_2$	-1	e_2e_3	e_1	$-\omega$	$-e_3$	e_1e_3
$\boldsymbol{e_3}$	e_3	$-e_1e_3$	$-e_2e_3$	-1	$-\omega$	e_1	e_2	e_1e_2
$\boldsymbol{e_1e_2}$	e_1e_2	e_2	$-e_1$	ω	-1	e_2e_3	$-e_1e_3$	$-e_3$
$\boldsymbol{e_1e_3}$	e_1e_3	e_3	ω	$-e_1$	$-e_2e_3$	-1	e_1e_2	$-e_2$
$\boldsymbol{e_2e_3}$	e_2e_3	$-\omega$	e_3	$-e_2$	e_1e_3	$-e_1e_2$	-1	e_1
$\boldsymbol{\omega}$	ω	e_2e_3	$-e_1e_3$	$-e_1e_2$	e_3	e_2	$-e_1$	1