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The Unified Theory of Complex and Hypercomplex Analytic Signals

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***Abstract*— The paper is devoted to the polar representation of *n*-D complex and hypercomplex analytic signals with emphasis on the 3-dimensional (3-D) case. Their definition is based on the proposed general form of the Cauchy integral. The definitions of complex/hypercomplex signals are presented in signal- and frequency domains. The new notion of lower rank signals is introduced. It is shown that starting with the 3-D analytic hypercomplex signals and decreasing their rank by extending the support in the frequency-space to a so called space quadrant, we get a signal having the quaternionic structure. The advantage of this procedure is demonstrated in the context of the polar representation of 3-D hypercomplex signals. Some new reconstruction formulas are presented. Their validation has been confirmed using two 3-D test signals: a Gaussian signal and a spherical signal.**

***Keywords*—complex/hypercomplex analytic signal, hypercomplex Fourier transform, hypercomplex delta distribution, polar representation**

Introduction

The theory of hypercomplex signals is a subject of many publications involving either mathematicians or engineers working in different fields [1]-[4]. It is based on the theory of hypercomplex numbers belonging to different algebras, e.g., to the Cayley-Dickson [5] or Clifford algebras [6]. Different definitions of complex/hypercomplex signals have appeared recently. The most interesting approach is the Clifford hypercomplex signal defined by Bülow and Sommer in [7]. For n = 2, it is identical to the quaternionic analytic signal A very detailed comparison of different definitions has been presented in [9]. Especially the case of 2-D analytic signals has been studied in detail and formulas relating analytic, quaternionic and monogenic signals have been derived.

This paper is devoted to the study and comparison of properties of n-D complex/hypercomplex signals with emphasis on the 3-D case. Especially, it will be shown that there are closed formulae enabling calculation of hypercomplex amplitude and phase functions in terms of the corresponding polar representation of complex functions. As well, it is noticed that the total number of amplitudes and phases of n-D complex and hypercomplex signals is equal to 2^n.

The Complex and Hypercomplex Multidimensional Analytic Functions Defined by the Cauchy Integral

Consider the n-D hypercomplex space $\mathbb{C}^n$ of hypercomplex variables: $\boldsymbol{z}=(z_1,z_2,\ldots,z_n): z_k = x_k + e_k y_k$ where e_k are imaginary units (in the domain of complex numbers they are usually denoted as $z_k = x_k + jy_k$). The space $\mathbb{C}^n$ is a Cartesian product of complex planes $\mathbb{C}_k$, $k=1,2,\ldots,n$, that is, $\mathbb{C}^n = \mathbb{C}_1 \times \mathbb{C}_2 \times \ldots \times \mathbb{C}_n$. We define a complex-valued n-D function $f(\boldsymbol{z})$, analytic (holomorphic) in the interior of a region $D^n = D_1 \times D_2 \times \ldots \times D_n$, $D^n \subset \mathbb{C}^n$, $D_k \subset \mathbb{C}_k$.

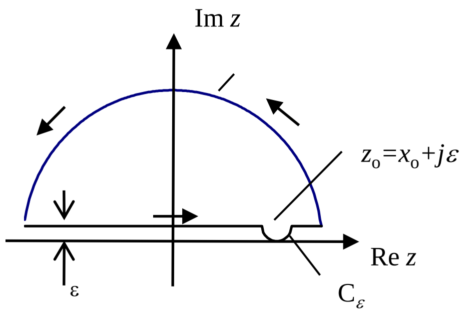

Fig. 1 The closed contour ∂D of integration in the complex plane (n = 1)

In [10], it has been shown that n-D analytic signals with single-orthant spectra are boundary distributions of n-D analytic functions. represented by the n-D Cauchy integral. In this paper, we propose the unified representation of complex and hypercomplex analytic signals introducing the *generalized form of the Cauchy integral*:

$$f(z) = \frac{1}{(2\pi e_1)(2\pi e_2)\dots(2\pi e_n)} \oint_{\partial D_1} \dots \oint_{\partial D_n} \frac{f(\xi_1, \dots, \xi_n)\, d\boldsymbol{\xi}_n}{(\xi_1 - z_1)\dots(\xi_n - z_n)}, \tag{1}$$

where ∂D_k are closed contours in D_k (see Fig. 1 for n=1). For n = 1, inserting $e_1 = j$, $z_1 = z$ and $\partial D_1 = \partial D$ we obtain the well known Cauchy integral

$$f(z) = \frac{1}{2\pi j} \oint_{\partial D} \frac{f(\xi)\, d\xi}{\xi - z}. \tag{2}$$

In the complex case [9], all imaginary units in (1) are equal and usually denoted with j and any order of integration can be applied. In the general case, if $\{e_k\}$ form the basis of a non-commutative algebra, the order of integration should be defined. It can be shown by induction that if (1) is valid for n-1, it is also valid for n variables [11]. Therefore, starting from (2) we can confirm the validity of (1).

The Complex and Hypercomplex Signals as Boundary Distributions of Analytic Functions

It has been shown in [10] that the successive integration of the classical Cauchy integral yields the following equivalent two forms of the n-D analytic signal:

$$\psi_c(\boldsymbol{x}) = \frac{1}{2^n} \prod_{k=1}^{n} \left(\mathbf{I}_k \{u(x_k)\} + j\mathbf{H}_k \{u(x_k)\} \right), \tag{3}$$

$$\psi_c(\boldsymbol{x}) = \frac{1}{2^n} u(\boldsymbol{x}) * \prod_{k=1}^{n} \left[\delta(x_k) + j \frac{1}{\pi x_k} \right], \tag{4}$$

where $\mathbf{I}_k$ is the 1-D identity operator w.r.t. x_k:

$$\mathbf{I}_k\left\{u\left(x_k\right)\right\}=u\left(x_k\right)*\delta\left(x_k\right)=u\left(x_k\right) \tag{5}$$

and $\mathbf{H}_k$ is the 1-D Hilbert transformation operator w.r.t. x_k:

$$\mathbf{H}_k\left\{u\left(x_k\right)\right\}=u\left(x_k\right)*\frac{1}{\pi x_k}=v_k\left(x\right). \tag{6}$$

Let us note that using the factor $1/2^n$ in (3) and (4) is a matter of convention. It is used in order to normalize the energy of a signal. In the following, we will omit it.

According to [10], the 1-D analytic signal $\psi_{\mathrm{c}}(x)=u(x)+jv(x)$ is a boundary distribution of the 1-D analytic function along the 0^+ side of the real axis of the $z=x+jy$ plane and has the form

$$\psi_{\mathrm{c}}\left(x\right)=\mathbf{I}\left\{u\right\}+j\mathbf{H}\left\{u\right\}=u\left(x\right)*\left[\delta\left(x\right)+j\frac{1}{\pi x}\right]. \tag{7}$$

For n = 2, we have

$$\psi_{\mathrm{c}}\left(x_1,x_2\right)=\mathbf{I}\left\{u\right\}-\mathbf{H}\left\{u\right\}+j\left(\mathbf{H_1}\left\{u\right\}+\mathbf{H_2}\left\{u\right\}\right), \tag{8}$$

which can be written as a product:

$$\psi_{\mathrm{c}}\left(x_1,x_2\right)=\left(\mathbf{I_1}\left\{u\right\}+j\mathbf{H_1}\left\{u\right\}\right)\left(\mathbf{I_2}\left\{u\right\}+j\mathbf{H_2}\left\{u\right\}\right). \tag{9}$$

The straightforward generalization of (3) for n-D hypercomplex signals as boundary distrbutions of (1) is

$$\psi_{\mathrm{c}}\left(\boldsymbol{x}\right)=\prod_{k=1}^{n}\left(\mathbf{I}_k\left\{u\left(x_k\right)\right\}+e_k\mathbf{H}_k\left\{u\left(x_k\right)\right\}\right). \tag{10}$$

We also have

$$\psi_{\mathrm{c}}\left(\boldsymbol{x}\right)=u\left(\boldsymbol{x}\right)*\Psi^{\delta}\left(\boldsymbol{x}\right) \tag{11}$$

where

$$\Psi^{\delta}\left(\boldsymbol{x}\right)=\prod_{k=1}^{n}\left[\delta\left(x_k\right)+e_k/\pi x_k\right] \tag{12}$$

is called the n-D hypercomplex delta distribution [12, 13].

The Fourier Spectral Representation of Multidimensional Analytic Signals

A. Complex and Hypercomplex Fourier Transforms

The most common method of analysis of the frequency content of a n-D real signal $u(\boldsymbol{x})$ is the classical n-D Fourier transform defined as

$$U\left(\boldsymbol{f}\right)=\int_{\mathbb{R}^n}u\left(\boldsymbol{x}\right)\prod_{i=1}^{n}\exp\left(-j2\pi f_i x_i\right)d^n\boldsymbol{x}\,. \tag{13}$$

Its generalization for the Clifford algebras has been proposed in [7], [14] and [15] and called the n-D Clifford Fourier transform given by

$$U_{Cl}\left(\boldsymbol{f}\right)=\int_{\mathbb{R}^n} u\left(\boldsymbol{x}\right)\prod_{i=1}^{n}\exp\left(-e_i 2\pi f_i x_i\right)d^n\boldsymbol{x} \tag{14}$$

where e_1, e_2, ..., e_n are elements of the Clifford algebra basis: $\left\{e_{i_1}e_{i_2}\ldots e_{i_k}:1\le i_1<i_2<\ldots<i_k\le n, 0\le k\le n\right\}$. The Clifford algebra is n^{th}-order associative and non-commutative algebra over $\mathbb{R}$ with $e_i^2=1$ or $e_i^2=-1$ and $e_ie_j=-e_je_i$ for $i\neq j$. Usually it is denoted with $Cl_{p,q}\left(\mathbb{R}\right)$ where p is a number of basis elements such as $e_i^2=1$ and q - the number of elements of the basis such that $e_i^2=-1$. It can be easily shown that $Cl_{0,2}\left(\mathbb{R}\right)$ is equivalent to the algebra of quaternions $\mathbb{H}$ (the 4^{th} order Cayley-Dickson algebra) [5]. In consequence, the definitions (13) and (14) are equivalent for n=1 and 2. If n = 2, (14) is called the (Left) Quaternionic Fourier transform.

In [13], the new definition of another hypercomplex Fourier transform inspired by the Cayley-Dickson algebra multiplication rules has been introduced. It has the form

$$U_{CD}\left(\boldsymbol{f}\right)=\int_{\mathbb{R}^n} u\left(\boldsymbol{x}\right)\prod_{i=1}^{n}\exp\left(-e_k 2\pi f_i x_i\right)d^n\boldsymbol{x} \tag{15}$$

with $k=2^{i-1}$. We see that (15) differs from (14) by the order of imaginary units e_i in exponents. For n = 3, (15) gets the form

$$U_{CD}\left(\boldsymbol{f}\right)=\int_{\mathbb{R}^3} u\left(\boldsymbol{x}\right)e^{-e_1 2\pi f_1 x_1}e^{-e_2 2\pi f_2 x_2}e^{-e_4 2\pi f_3 x_3}d^3\boldsymbol{x}\,. \tag{16}$$

B. Frequency-domain Definitions of Complex and Hypercomplex Analytic Signals

Our research is concentrated on signals with a single orthant spectra in the n-D frequency space. An orthant is a half-axis in 1-D, a single quadrant in 2-D, an octant in 3-D, etc. Let us recall the definition of the n-D unit step operator $\mathbf{1}\left(\boldsymbol{f}\right)$:

$$\mathbf{1}\left(\boldsymbol{f}\right)=\prod_{i=1}^{n}\left(0.5+0.5\,\mathrm{sgn}\left(f_i\right)\right). \tag{17}$$

The (complex) analytic signal $\psi_{\mathrm{c}}\left(\boldsymbol{x}\right)$ has been defined in [16] as the inverse Fourier transform of a single-orthant spectrum:

$$\psi_{\mathrm{c}}\left(\boldsymbol{x}\right)=\mathbb{F}^{-1}\left\{\mathbf{1}\left(\boldsymbol{f}\right)\cdot U\left(\boldsymbol{f}\right)\right\}, \tag{18}$$

where $U(\boldsymbol{f})$ is given by (13). Analogously, the *n*-D Clifford analytic signal $\psi_{Cl}(\boldsymbol{x})$ with a single-orthant spectrum [7] has the form

$$\psi_{Cl}(\boldsymbol{x}) = \mathbb{F}_{Cl}^{-1}\{\mathbf{1}(\boldsymbol{f})\cdot U_{Cl}(\boldsymbol{f})\}, \tag{19}$$

where $\mathbb{F}_{Cl}^{-1}$ is the inverse Clifford Fourier transform [7], [14], [15]. It has been shown in [13] that in 3-D, the signal (19) has a split-biquaternionic structure. Also in [13], we introduced the frequency-domain definition of a new *n*-D hypercomplex signal with a single-orthant spectrum as follows:

$$\psi_{\mathrm{h}}(\boldsymbol{x}) = \mathbb{F}_{CD}^{-1}\{\mathbf{1}(\boldsymbol{f})\cdot U_{CD}(\boldsymbol{f})\}. \tag{20}$$

The 3-D hypercomplex signal $\psi_{\mathrm{h}}(x_1, x_2, x_3)$ defined in (20) has the form of an octonionic signal. Let us note that, either (19) or (20), are 8-D hypercomplex representations of a 3-D real signal. However, only the expression (20) represents a signal with a well defined norm (the standard Euclidean norm) equal to the norm of the Hahn's analytic signal [16]. The norm of the Clifford analytic signal is a semi-norm given by the formula $u^2 + v_1^2 + v_2^2 + v_3^3 + v_{12}^2 + v_{13}^2 + v_{23}^2 - v^2$. Due to the minus sign of the last term the square root of this norm (defining the amplitude of the 3-D hypercomplex signal) is a complex function. Therefore, the amplitude of the 3-D hypercomplex signal is undefined.

The Notion of Ranking of 3-D Analytic Signals

To get a better understanding of the polar representation of 3-D (or even higher dimensional) analytic signals we introduce the notion of ranking. We assign the highest *rank* equal *n* to signals with single-orthant spectra. The lower *rank* equal (*n-1*) is assigned to a union of two signals of *rank n*. The support of the spectrum is doubled. The *rank* equal *1* is assigned to signals with half-space spectrum, a union of n/2 orthants.

In this paper, we consider the signals with the spectrum support limited to the half-space $f_1 > 0$. For example, 3-D analytic signals are defined by the inverse Fourier transform of the complex or hypercomplex single-octant spectra [16], [17]. The half-space is a union of four separate octants. Therefore, it is possible to define four different complex signals with single-octant spectra:

$$\psi_{\mathrm{c}}^{(1)} = u - v_{12} - v_{13} - v_{23} + j(v_1 + v_2 + v_3 - v), \tag{21}$$

$$\psi_{\mathrm{c}}^{(3)} = u + v_{12} - v_{13} + v_{23} + j(v_1 - v_2 + v_3 + v), \tag{22}$$

$$\psi_{\mathrm{c}}^{(5)} = u - v_{12} + v_{13} + v_{23} + j\left(v_1 + v_2 - v_3 + v\right), \tag{23}$$

$$\psi_{\mathrm{c}}^{(7)} = u + v_{12} + v_{13} - v_{23} + j\left(v_1 - v_2 - v_3 - v\right) \tag{24}$$

with spectral support in octants No. 1, 3, 5 and 7 (superscript indicates the octant's label). We assign to signals (21)-(24) the highest *rank 3*. The signal of a lower *rank 2* is obtained as the sum of two signals of *rank 3* with spectra limited to two adjacent octants forming the so called *space quadrant*. Adding (21) and (23) we get

$$\psi_{\mathrm{c}}^{(1,5)} = \frac{\psi_{\mathrm{c}}^{(1)} + \psi_{\mathrm{c}}^{(5)}}{2} = u - v_{12} + j\left(v_1 + v_2\right). \tag{25}$$

Similarly, for (22) and (24) we have

$$\psi_{\mathrm{c}}^{(3,7)} = \frac{\psi_{\mathrm{c}}^{(3)} + \psi_{\mathrm{c}}^{(7)}}{2} = u + v_{12} + j\left(v_1 - v_2\right). \tag{26}$$

The addition of (25) and (26) gives the signal of *rank 1* with the spectrum in the half-space:

$$\psi_{\mathrm{c}}^{(1,3,5,7)} = \frac{\psi_{\mathrm{c}}^{(1)} + \psi_{\mathrm{c}}^{(3)} + \psi_{\mathrm{c}}^{(5)} + \psi_{\mathrm{c}}^{(7)}}{4} = u + jv_1. \tag{27}$$

In the same way, basing on (20), we define four hypercomplex analytic signals of *rank 3* with spectra in octants No. 1, 3, 5 and 7:

$$\begin{aligned}\psi_{\mathrm{h}}^{(1)} &= u + e_1 v_1 + e_2 v_2 + e_3 v_{12} + e_4 v_3 + e_5 v_{13} + e_6 v_{23} + e_7 v = \\ &= u + e_1 v_1 + e_2 v_2 + e_3 v_{12} + \left(v_3 + e_1 v_{13} + e_2 v_{23} + e_3 v\right) e_4,\end{aligned} \tag{28}$$

$$\psi_{\mathrm{h}}^{(3)} = u + e_1 v_1 - e_2 v_2 - e_3 v_{12} + e_4 v_3 + e_5 v_{13} - e_6 v_{23} - e_7 v, \tag{29}$$

$$\begin{aligned}\psi_{\mathrm{h}}^{(5)} &= u + e_1 v_1 + e_2 v_2 + e_3 v_{12} - e_4 v_3 - e_5 v_{13} - e_6 v_{23} - e_7 v = \\ &= u + e_1 v_1 + e_2 v_2 + e_3 v_{12} - \left(v_3 + e_1 v_{13} + e_2 v_{23} + e_3 v\right) e_4,\end{aligned} \tag{30}$$

$$\psi_{\mathrm{h}}^{(7)} = u + e_1 v_1 - e_2 v_2 - e_3 v_{12} - e_4 v_3 - e_5 v_{13} + e_6 v_{23} + e_7 v. \tag{31}$$

In the second line of (28) and (30) we presented the octonionic signal as a complex sum of two quaternionic signals. The same applies for (29) and (31).

The hypercomplex signals of *rank 2* are defined similarly as in (25) and (26):

$$\psi_{\mathrm{h}}^{(1,5)} = \frac{\psi_{\mathrm{h}}^{(1)} + \psi_{\mathrm{h}}^{(5)}}{2} = u + e_1 v_1 + e_2 v_2 + e_3 v_{12}, \tag{32}$$

$$\psi_{\mathrm{h}}^{(3,7)} = \frac{\psi_{\mathrm{h}}^{(3)} + \psi_{\mathrm{h}}^{(7)}}{2} = u + e_1 v_1 - e_2 v_2 - e_3 v_{12} \tag{33}$$

where u, v_1, v_2 and v_{12} are 3-D real functions. Note that (32) has the form of the first quaternion in (28) and the same applies for (33). The hypercomplex signal of *rank 1* with the spectrum in the half-space is defined as the sum of signals (32) and (33):

$$\psi_{\mathrm{h}}^{(1,3,5,7)} = \frac{\psi_{\mathrm{h}}^{(1)} + \psi_{\mathrm{h}}^{(3)} + \psi_{\mathrm{h}}^{(5)} + \psi_{\mathrm{h}}^{(7)}}{4} = u + e_1 v_1 . \tag{34}$$

We observe that the *3-D rank 1* complex and hypercomplex analytic signals have the same form (compare with (27)).

The Polar Representation of 3-D Hypercomplex Analytic Signals

In order to define the polar form in complex/hypercomplex case, it is necessary to specify how many amplitudes and phase functions are necessary to define uniquely a signal in 3-D. Let us start with the Lemma 1:

Lemma 1: The total number of amplitude and phase functions of *n*-D complex/hypercomplex analytic signals equals $M = 2^n$.

In Table 1, the number of amplitudes and phases of complex and hypercomplex analytic signals for n = 1, 2, 3 and 4 is given.

Table 1. The number of amplitudes and phase of *n*-D complex/hypercomplex analytic signals, n = 1, 2, 3, 4

<table>
<tr><th rowspan="2">n</th><th rowspan="2">type</th><th colspan="3">Number of:</th></tr>
<tr><th>amplitudes</th><th>phases</th><th>Total M</th></tr>
<tr><td rowspan="2">1</td><td>complex</td><td rowspan="2">1</td><td rowspan="2">1</td><td rowspan="2">2</td></tr>
<tr><td>hypercomplex</td></tr>
<tr><td rowspan="2">2</td><td>complex</td><td>2</td><td>2</td><td rowspan="2">4</td></tr>
<tr><td>hypercomplex</td><td>1</td><td>3</td></tr>
<tr><td rowspan="2">3</td><td>complex</td><td>4</td><td>4</td><td rowspan="2">8</td></tr>
<tr><td>hypercomplex</td><td>1</td><td>7</td></tr>
<tr><td rowspan="2">4</td><td>complex</td><td>8</td><td>8</td><td rowspan="2">16</td></tr>
<tr><td>hypercomplex</td><td>1</td><td>15</td></tr>
</table>

In case of separable signals (products of 1-D signals), for n=2: we have $M = 3 = 1 + 2$ (one amplitude and 2 phases) and for n =3: $M = 4 = 1 + 3$ (one amplitude and three phases).

A. The polar form of 3-D complex analytic signals

The 3-D complex signals (21)-(24) with a single-octant spectra can be written in their polar forms [16] as follows:

$$\psi_{\mathrm{c}}^{(1)} = A_{\mathrm{c}}^{(1)} e^{j\varphi_{\mathrm{c}}^{(1)}}, \tag{35}$$

$$\psi_{\mathrm{c}}^{(3)} = A_{\mathrm{c}}^{(3)} e^{j\varphi_{\mathrm{c}}^{(3)}}, \tag{36}$$

$$\psi_{\mathrm{c}}^{(5)} = A_{\mathrm{c}}^{(5)} e^{j\varphi_{\mathrm{c}}^{(5)}}, \tag{37}$$

$$\psi_{\mathrm{c}}^{(7)} = A_{\mathrm{c}}^{(7)} e^{j\varphi_{\mathrm{c}}^{(7)}} \tag{38}$$

where $A_{\mathrm{c}}^{(i)}$ and $\varphi_{\mathrm{c}}^{(i)}$ are respectively the amplitude- and phase functions. The superscript (i) denotes the label of the octant, i = 1, 3, 5, 7. Let us recall that, for separable 3-D signals, all the four amplitudes $A_{\mathrm{c}}^{(i)}$ are the same and equal to

$$A_0 = \sqrt{u^2 + v_1^2 + v_2^2 + v_{12}^2 + v_3^2 + v_{13}^2 + v_{23}^2 + v^2} \tag{39}$$

and the four phase functions $\varphi_{\mathrm{c}}^{(i)}$ are linear combinations of 1-D phase functions $\alpha_i(x_i), i = 1, 2, 3$:

$$\varphi_{\mathrm{c}}^{(1)}(x_1, x_2, x_3) = \alpha_1(x_1) + \alpha_2(x_2) + \alpha_3(x_3), \tag{40}$$

$$\varphi_{\mathrm{c}}^{(3)}(x_1, x_2, x_3) = \alpha_1(x_1) - \alpha_2(x_2) + \alpha_3(x_3), \tag{41}$$

$$\varphi_{\mathrm{c}}^{(5)}(x_1, x_2, x_3) = \alpha_1(x_1) + \alpha_2(x_2) - \alpha_3(x_3), \tag{42}$$

$$\varphi_{\mathrm{c}}^{(7)}(x_1, x_2, x_3) = \alpha_1(x_1) - \alpha_2(x_2) - \alpha_3(x_3). \tag{43}$$

Let us introduce the polar forms of the *3-D rank 2* complex analytic signals defined in (25) and (26):

$$\psi_{\mathrm{c}}^{(1,5)} = A_{\mathrm{c}}^{(1,5)} e^{j\varphi_{\mathrm{c}}^{(1,5)}}, \tag{44}$$

$$\psi_{\mathrm{c}}^{(3,7)} = A_{\mathrm{c}}^{(3,7)} e^{j\varphi_{\mathrm{c}}^{(3,7)}}. \tag{45}$$

For *3-D rank 2* hypercomplex signals having the quaternionic structure as shown in (32) and (33) we can write (compare with (A3), Appendix A)

$$\psi_{\mathrm{h}}^{(1,5)} = A_{\mathrm{h}}^{(1,5)} e^{e_1\phi_1^{(1,5)}} e^{e_3\phi_3^{(1,5)}} e^{e_2\phi_2^{(1,5)}}, \tag{46}$$

$$\psi_{\mathrm{h}}^{(3,7)} = A_{\mathrm{h}}^{(3,7)} e^{e_1\phi_1^{(3,7)}} e^{-e_3\phi_3^{(3,7)}} e^{-e_2\phi_2^{(3,7)}} \tag{47}$$

where

$$A_{\mathrm{h}}^{(1,5)} = A_{\mathrm{h}}^{(3,7)} = \sqrt{u^2 + v_1^2 + v_2^2 + v_{12}^2} \tag{48}$$

and

$$\mathrm{tg}\left(2\varphi_1^{(1,5)}\right) = \frac{2\left(uv_1 + v_2 v_{12}\right)}{u^2 - v_1^2 + v_2^2 - v_{12}^2}, \tag{49}$$

$$\mathrm{tg}\left(2\varphi_2^{(1,5)}\right) = \frac{2\left(uv_2 + v_1 v_{12}\right)}{u^2 + v_1^2 - v_2^2 - v_{12}^2}, \tag{50}$$

$$\sin\left(2\varphi_3^{(1,5)}\right) = \frac{\left(A_{\mathrm{c}}^{(1,5)}\right)^2 - \left(A_{\mathrm{c}}^{(3,7)}\right)^2}{\left(A_{\mathrm{c}}^{(1,5)}\right)^2 + \left(A_{\mathrm{c}}^{(3,7)}\right)^2}. \tag{51}$$

Similarly to (A5) and (A6) (Appendix A), we can write the relations between the phase functions of the 3-D *rank 2* complex and hypercomplex signals. We have

$$\varphi_1^{(1,5)} = 0.5\left(\varphi_{\mathrm{c}}^{(1,5)} + \varphi_{\mathrm{c}}^{(3,7)}\right) \tag{52}$$

and

$$\varphi_1^{(1,5)} = 0.5\left(\varphi_{\mathrm{c}}^{(1,5)} + \varphi_{\mathrm{c}}^{(3,7)}\right). \tag{53}$$

We see that the amplitude and three phase functions of the *3-D rank 2* hypercomplex signal are expressed as functions of two amplitudes and two phase functions of the *rank 2* complex (analytic) signal.

B. The polar form of 3-D hypercomplex analytic signals

Let us recall that the polar form of the quaternionic analytic signal is derived using the Euler angles (or Rodrigues matrix). We have not found any paper describing a similar derivation for 3-D hypercomplex signals. There are suggestions, that the theory of Lie groups should be applied or eventually a so called SO4 group. At present we failed to find a solution. Let us say that we consulted the problem with some experts. Nobody was able to give a suggestion about possible solution. In order to find it using a method of deduction we proposed the following polar form of the octonionic signal:

$$\begin{aligned}\psi_{\mathrm{h}}\left(x_1, x_2, x_3\right) = \\ A_0 \exp\left(e_1\varphi_1\right)\exp\left(e_3\varphi_3\right)\exp\left(e_2\varphi_2\right)\exp\left(e_7\varphi_7\right)\exp\left(e_4\varphi_4\right)\exp\left(e_6\varphi_6\right)\exp\left(e_5\varphi_5\right)\end{aligned} \tag{54}$$

where the amplitude A_0 is given by (39) and seven phase functions $\varphi_i, i = 1, \ldots, 7$ have been defined in analogy to the polar representation of the 2-D quaternionic signal (A3) (Appendix

A). They form two groups: the first one: $\{\varphi_1,\varphi_2,\varphi_4,\varphi_5\}$ is defined by the four phase functions $\varphi_{\mathrm{c}}^{(i)}$ of the complex analytic signal (see Eqs. (35)-(38):

$$\varphi_1 = \frac{1}{4}\left(\varphi_{\mathrm{c}}^{(1)} + \varphi_{\mathrm{c}}^{(3)} + \varphi_{\mathrm{c}}^{(5)} + \varphi_{\mathrm{c}}^{(7)}\right), \tag{55}$$

$$\varphi_2 = \frac{1}{4}\left(\varphi_{\mathrm{c}}^{(1)} + \varphi_{\mathrm{c}}^{(3)} - \varphi_{\mathrm{c}}^{(5)} - \varphi_{\mathrm{c}}^{(7)}\right), \tag{56}$$

$$\varphi_4 = \frac{1}{4}\left(\varphi_{\mathrm{c}}^{(1)} - \varphi_{\mathrm{c}}^{(3)} + \varphi_{\mathrm{c}}^{(5)} - \varphi_{\mathrm{c}}^{(7)}\right), \tag{57}$$

$$\varphi_5 = \frac{1}{4}\left(\varphi_{\mathrm{c}}^{(1)} - \varphi_{\mathrm{c}}^{(3)} - \varphi_{\mathrm{c}}^{(5)} + \varphi_{\mathrm{c}}^{(7)}\right). \tag{58}$$

The second group: $\{\varphi_3,\varphi_6,\varphi_7\}$ is defined by the corresponding four amplitude functions:

$$\sin\left(4\varphi_3\right) = \frac{\left(A_{\mathrm{c}}^{(1)}\right)^2 - \left(A_{\mathrm{c}}^{(3)}\right)^2}{\left(A_{\mathrm{c}}^{(1)}\right)^2 + \left(A_{\mathrm{c}}^{(3)}\right)^2}, \tag{59}$$

$$\sin\left(4\varphi_6\right) = \frac{\left(A_{\mathrm{c}}^{(5)}\right)^2 - \left(A_{\mathrm{c}}^{(7)}\right)^2}{\left(A_{\mathrm{c}}^{(5)}\right)^2 + \left(A_{\mathrm{c}}^{(7)}\right)^2}, \tag{60}$$

$$\sin\left(4\varphi_7\right) = \frac{\left(A_{\mathrm{c}}^{(1)}\right)^2 + \left(A_{\mathrm{c}}^{(3)}\right)^2 - \left(A_{\mathrm{c}}^{(5)}\right)^2 - \left(A_{\mathrm{c}}^{(7)}\right)^2}{\left(A_{\mathrm{c}}^{(1)}\right)^2 + \left(A_{\mathrm{c}}^{(3)}\right)^2 + \left(A_{\mathrm{c}}^{(5)}\right)^2 + \left(A_{\mathrm{c}}^{(7)}\right)^2}. \tag{61}$$

The forms (59)-(61) have been deduced from the 2-D case (see Appendix A, Eq. (A5)).

C. The verification of the Eqs.(54) to (61).

A real signal $u(\boldsymbol{x})$ can be reconstructed from its polar representation of the corresponding analytic signals using reconstruction formulae. For 1-D signals, we have $u_{\mathrm{c}}(x)=A\cos(\varphi)$. For 2-D complex signals we have

$$u_{\mathrm{c}}\left(x_1,x_2\right) = \frac{A_{\mathrm{c}}^{(1)}\cos\left(\varphi^{(1)}\right) + A_{\mathrm{c}}^{(3)}\cos\left(\varphi^{(3)}\right)}{2} \tag{62}$$

and for the 2-D hypercomplex (quaternionic) signals we have

$$u_{\mathrm{h}}\left(x_1,x_2\right) = A_0\left[\cos\left(\phi_1\right)\cos\left(\phi_2\right)\cos\left(\phi_3\right) + \sin\left(\phi_1\right)\sin\left(\phi_2\right)\sin\left(\phi_3\right)\right] \tag{63}$$

The same form applies for the 3-D rank-2 hypercomplex signals {all functions are 3-D):

$$u_{\mathrm{h}}\left(x_1,x_2,x_3\right) = A_0\left[\cos\left(\phi_1\right)\cos\left(\phi_2\right)\cos\left(\phi_3\right) + \sin\left(\phi_1\right)\sin\left(\phi_2\right)\sin\left(\phi_3\right)\right] \tag{64}$$

The 3-D complex signal is reconstructed using

$$u_{\mathrm{c}}\left(x_1, x_2, x_3\right)=\frac{A_{\mathrm{c}}^{(1)} \cos \varphi_{\mathrm{c}}^{(1)}+A_{\mathrm{c}}^{(3)} \cos \varphi_{\mathrm{c}}^{(3)}+A_{\mathrm{c}}^{(5)} \cos \varphi_{\mathrm{c}}^{(5)}+A_{\mathrm{c}}^{(7)} \cos \varphi_{\mathrm{c}}^{(7)}}{4} \tag{65}$$

Let us note that reconstructed complex signals will be denoted with the subscript “c”, while the reconstructed hypercomplex signals with “h”.

Let us continue the discussion formulating next two lemmas:

Lemma 2: A *n*-D nonseparable real signal can be reconstructed from a polar form of $M=2^{n-1}$ different complex signals with single-orthant spectra (total support in the half-space).

Lemma 3: A *n*-D nonseparable real signal can be reconstructed from a polar form of $M/2=2^{n-2}$ different hypercomplex signals with single-orthant spectra (total support in the quarter-space).

The Lemma 2 needs no evidence. Eqs. (62) and (64) satisfy it and the extension for $n > 3$ is straightforward. Differently, we only know that the Lemma 3 is satisfied for $n = 2$ (quaternionic signals). Moreover, it states that the 3-D real signal cannot be reconstructed using the single amplitude and the seven phase functions of the signal (54) of single-octant spectral support (1/8 of the total space). However, the requirement of the ¼ spectral support is satisfied by the *rank 2* hypercomplex signal defined by (32). Its polar form is similar as defined in [7] for 2-D quaternionic signals:

$$\psi_{\mathrm{h}}^{(1,5)}=A_0^{(1,5)} e^{e_1 \varphi_1^{(1,5)}} e^{e_3 \varphi_3^{(1,5)}} e^{e_2 \varphi_2^{(1,5)}} \tag{66}$$

where the amplitude

$$A_0^{(1,5)}=\sqrt{u^2+v_1^2+v_2^2+v_{12}^2} \tag{67}$$

and three phase angles $\varphi_i^{(1,5)},\ i=1,2,3$ can be calculated using two amplitudes and two phase functions of the 3-D *rank 2* complex signals (see (51)-(53)).

Verification of the Reconstruction Formulae using Numerical Calculations

Due to the lack of theoretical derivation of the polar form (54), we applied numerical calculations using a Gaussian 3-D real test signal (see Appendix 2). Using the four amplitudes and four phase functions defined by (35) to (38) (Cartesian form is defined by (21) to 24)) we calculated the amplitude using (39) and the seven phase functions given by (55)-(61).

The reconstructed signal is calculated using the formula

$$\begin{aligned} u_{\mathrm{h}}\left(x_1,x_2,x_3\right) = A_0[& c_1c_2c_3c_4c_5c_6c_7 + s_1s_2s_3c_4c_5c_6c_7 - s_1c_2c_3s_4s_5c_6c_7 + c_1s_2s_3s_4s_5c_6c_7 - s_1s_2c_3s_4c_5s_6c_7 \\ & + s_1c_2s_3s_4c_5s_6c_7 - c_1c_2s_3c_4s_5s_6c_7 - s_1s_2c_3c_4s_5s_6c_7 + c_1c_2s_3s_4c_5c_6s_7 + s_1s_2c_3s_4c_5c_6s_7 \\ & + c_1s_2c_3c_4s_5c_6s_7 - s_1c_2s_3c_4s_5c_6s_7 - s_1c_2c_3c_4c_5s_6s_7 + c_1s_2s_3c_4c_5s_6s_7 - c_1c_2c_3s_4s_5s_6s_7 \\ & - s_1s_2s_3s_4s_5s_6s_7] \end{aligned} \tag{68}$$

where $c_i = \cos\left(\varphi_i\right)$ and $s_i = \sin\left(\varphi_i\right)$.

A. N*umerical calculations using 3-D Gaussian test signal*

The original and reconstructed signals are displayed using the Matlab graphical editor. We show cross-sections with fixed values of the third variable x_3 = 0. The observation of the tables of 3-D data confirmed, that such a choice of a cross-sections is representative. We applied the following parameters of the Gaussian signal: $\sigma_1 = \sigma_2 = \sigma_3 = 0.5$ and $\rho_{12} = \rho_{13} = \rho_{23} = 0.7$ (nonseparable case) and $\rho_{12} = \rho_{13} = \rho_{23} = 0$ (separable case).

The accuracy of the numerical calculation has been verified by comparing the original real signal (Fig. 2a) with the reconstructed signal (65). The calculated difference equals zero (Fig. 2b). The same result has been obtained using the 3-D version of the reconstruction formulae (63) applied to the hypercomplex 3-D *rank 2* signal (32). Note that this signal satisfies the Lemma 3.

Finally, we calculated the reconstructed signal given by (68), i.e., using the single amplitude and seven phase functions of (54). Calculations have shown that the most dominating term is $A_0c_1c_2c_3c_4c_5c_6c_7$, and a few terms are very small w.r.t. the dominating term and the rest are completely insignificant. Fig. 3a shows the reconstructed signal. It differs only slightly from the original one of Fig. 2a. Fig. 3b shows the difference between the original and reconstructed signals. It is of the order of a few percents w.r.t. the original signal. We detected that the addition of other 15 terms to the dominating term increases the difference. Fig. 3c shows the difference between the the dominating term and the original signal. Differently to Fig. 3b, the difference is symmetric. Therefore, the original signal can be approximately reconstructed using the dominating term of (68) We may assume that the above calculations confirmed the Lemma 3 in 3-D case. However, only theorethical derivation of the polar form (54) can yield a more safe confirmation.

Let us have a comment about the deduced formulae (55)-(61). For separable signals, all four amplitudes of the 3-D complex signals are equal. In consequence the phase functions defined by (59)-(61). i.e., $\varphi_3 = \varphi_6 = \varphi_7 = 0$. The insertion of formulas (40)-(43) into (55)-(58) yields $\varphi_1 = \alpha_1$, $\varphi_2 = \alpha_3$, $\varphi_4 = \alpha_2$ and $\varphi_5 = 0$. In consequence, the polar form (54) reduces to

$$\psi_{\mathrm{h}}^{(1)}\left(x_1, x_2, x_3\right) = A_0 e^{e_1\alpha_1} e^{e_2\alpha_2} e^{e_4\alpha_3} \tag{69}$$

and has exactly the same form as the polar representation of separable complex signals. Both forms differ only by the imaginary units. This fact validates strongly the deduced Eqs. (55)-(58).

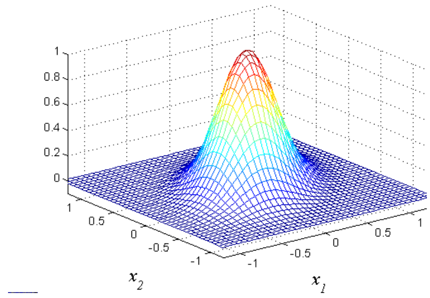

Fig. 2a. Cross-section $u(x_1, x_2, x_3 = 0)$ of the real Gaussian 3-D nonseparable signal. Exactly the same picture applies for reconstructed signals defined by (63)-(64) .

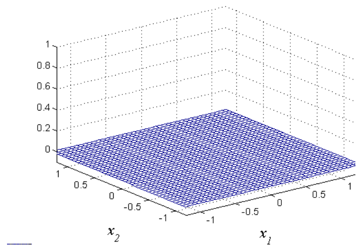

Fig. 2b. The difference between the nonseparable real 3-D Gaussian signal and its reconstructed versions defined by (64) (rank 2 hypercomplex) and (65) (complex). In both cases the reconstruction is perfect.

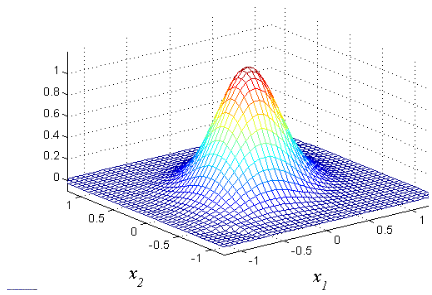

Fig. 3a. The signal reconstructed using (68). It differs only slightly from the original signal of Fig. 2a.

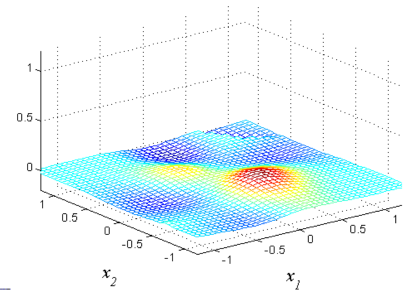

Fig.3b. The difference between the signal reconstructed using (68) and the original signal of Fig. 2a.

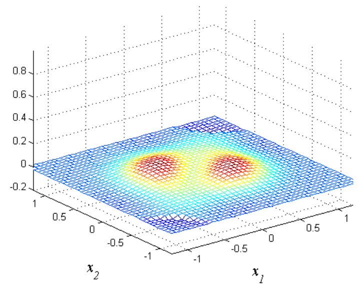

Fig. 3c. The difference between the signal reconstructed using the dominating first term of (68) and the original signal of Fig. 2a.

B. Example with a 3-D signal in the form of a sphere

In the above example, the 3-D signal defined by (B1) and the signal calculated numerically by the inverse Fourier transform of the spectrum given by (B4) (40 samples for each variable) have exactly the same shape. This is a feature of the smooth Gaussian signal. This is not a feature of signals with sharp edges, for example, the signal in the form of a sphere (see Appendix C). Fig. 4a shows the cross-section (x_3 = 0) of a sphere defined by (C1) and Fig. 4b the same defined by the inverse Fourier transform of the spectrum given by (C2). Their difference is shown in Fig. 4c. The signal reconstructed using (65) has the form shown in Fig.4b, i.e., the reconstruction is perfect only w.r.t. the signal defined by the inverse Fourier transform of the spectrum.

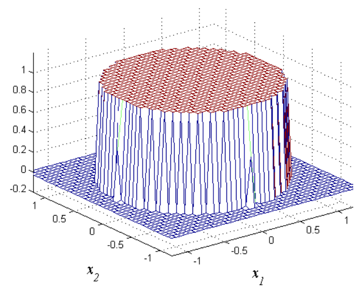

Fig.4a. The cross-section (x_3 = 0) of the sphere given by (C1).

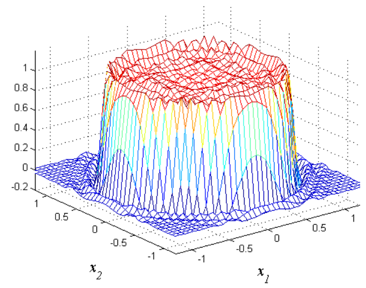

Fig.4b. The signal of Fig.4a obtained by the numerical calculation of the inverse Fourier transform of the spectrum given by (C2).

Finally, let us check the reconstruction formulae (68) for the sphere. Fig. 4d shows the reconstructed signal and Fig. 4e the difference between Figs 4b and 4d. We see that the reconstruction is imperfect (Lemma 3).

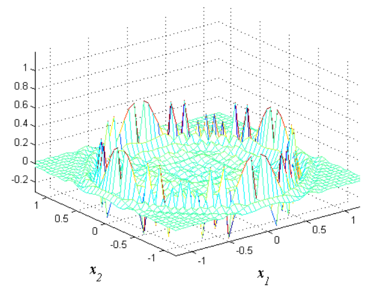

Fig.4c. The difference of the cross-sections displayed in Figs 4a and 4b.

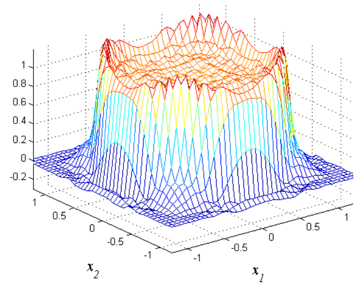

Fig.4d. The cross-section of the sphere reconstructed using (68)

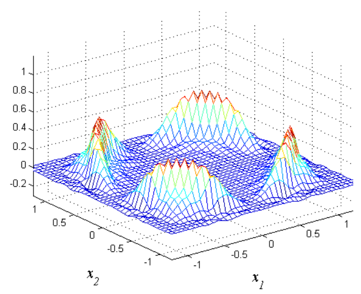

Fig. 4e. The difference of the cross-sections of Figs 4b and 4d. It confirms the Lemma 3.

Conclusions

(i) The generalized form of the Cauchy integral presented in this paper shows that n-D complex and hypercomplex analytic signals are both boundary distributions of the corresponding analytic functions. Therefore, complex and hypercomplex analytic signals have the same theoretical roots.

(ii) It has been shown, that the formulae defining the polar representation of 2-D and 3-D hypercomplex signals can be derived starting with the corresponding amplitude and phase functions of complex signals. We believe, that this procedure applies for n-D signals.

(iii) It has been recalled that n-D real signals can be reconstructed using the amplitude and phase functions of its polar form of $M = 2^{n-1}$ complex signals (Lemma 2) with the total spectral support in a half-space.

(iv) For 2-D and 3-D real signals, we have shown that they can be reconstructed using the amplitude and phase functions of $M = 2^{n-2}$ hypercomplex signals (Lemma 3) with the total support in a quarter of the space. We believe that lemma 3 is valid for n>3.

(ivi) The paper presents the notions of a lower rank complex and hypercomplex signals and shows that a real 3-D signal can be reconstructed using the polar form of a 3-D *rank 2* signal.

The paper has been illustrated with 3-D analytic signals with single-octant spectra and the corresponding octonionic 3-D signals with Cayley-Dickson algebra imaginary units. The polar representation of the octonionic signals is exactly known only for *rank 2* signals having the form of the quaternionic valued 3-D signal. The theoretical derivation of the polar representation of the *rank 3* octonionic signal is, to our knowledge, an unsolved problem. However, we presented a hypothesis that the polar form of the octonionic signal defines a single amplitude and seven phase functions. Theoretical results have been tested using the 3-D non-separable Gaussian signal. Next research is needed.

APPENDIX A. 2-D AND 3-D ANALYTIC SIGNALS

In 2-D, it is possible to define four analytic signals with spectra in successive quadrants of the frequency space. We have two conjugated pairs of signals: $\psi_1 = \psi_4^*$ and $\psi_3 = \psi_2^*$. The labelling of quadrants of the (f_1, f_2)-space is the same as in [12]. So, the 2-D real signal is represented by two different analytic signals: the first one with a the spectrum limited to the 1st quadrant (f_1>0, f_2>0) is

$$\psi_c^{(1)}\left(x_1, x_2\right) = u - v + j\left(v_1 + v_2\right) = A_c^{(1)}\left(x_1, x_2\right) e^{j\varphi_c^{(1)}\left(x_1, x_2\right)} \tag{A1}$$

and the second one with a spectrum in the 3rd quadrant ($f_1>0$, $f_2<0$):

$$\psi_{\mathrm{c}}^{(3)}\left(x_1,x_2\right)=u+v+j\left(v_1-v_2\right)=A_{\mathrm{c}}^{(3)}\left(x_1,x_2\right)e^{j\varphi_{\mathrm{c}}^{(3)}\left(x_1,x_2\right)}. \tag{A2}$$

The polar representation defines two different amplitudes $A^{(1)}$, $A^{(3)}$ and two different phase functions $\varphi^{(1)}$, $\varphi^{(3)}$. Two analytic signals can be equivalently replaced by a single quaternionic signal [7], [9] with the spectrum in the 1st quadrant of the frequency space:

$$\psi_{\mathrm{h}}^{(1)}\left(x_1,x_2\right)=u+v_1e_1+v_2e_2+ve_3=A_0\left(x_1,x_2\right)e^{e_1\phi_1}e^{e_3\phi_3}e^{e_2\phi_2} \tag{A3}$$

where

$$A_0^2=u^2+v_1^2+v_2^2+v^2=\frac{\left(A_{\mathrm{c}}^{(1)}\right)^2+\left(A_{\mathrm{c}}^{(3)}\right)^2}{2} \tag{A4}$$

and

$$\varphi_1=0.5\left(\varphi_{\mathrm{c}}^{(1)}+\varphi_{\mathrm{c}}^{(3)}\right), \tag{A5}$$

$$\varphi_2=0.5\left(\varphi_{\mathrm{c}}^{(1)}-\varphi_{\mathrm{c}}^{(3)}\right), \tag{A6}$$

$$\sin\left(2\varphi_3\right)=\frac{\left(A_{\mathrm{c}}^{(1)}\right)^2-\left(A_{\mathrm{c}}^{(3)}\right)^2}{\left(A_{\mathrm{c}}^{(1)}\right)^2+\left(A_{\mathrm{c}}^{(3)}\right)^2}. \tag{A7}$$

Concluding, there are close formulae enabling the calculation of the quaternionic polar representation using the analytic polar one. Note that in both cases, we deal with four different functions, i.e., $A_{\mathrm{c}}^{(1)}$, $A_{\mathrm{c}}^{(3)}$, $\varphi_{\mathrm{c}}^{(1)}$, $\varphi_{\mathrm{c}}^{(3)}$ (analytic case) correspond to A_0, φ_1, φ_2, φ_3 (quaternionic case) (see Table 1)

APPENDIX B. THE 3-D GAUSSIAN SIGNAL

The 3-D Gaussian signal is defined by

$$u\left(x_1,x_2,x_3\right)=\left(2\pi\right)^{-3/2}\left|M\right|^{-1/2}\exp\left\{\frac{-1}{2\left|M\right|}\sum_{i,j=1}^{3}\left|M_{ij}\right|x_ix_j\right\} \tag{B1}$$

where

$$\left|M\right|=\sigma_1^2\sigma_2^2\sigma_3^2\left(1+\rho_{12}\rho_{23}\rho_{13}+\rho_{13}\rho_{12}\rho_{23}-\rho_{12}^2-\rho_{23}^2-\rho_{13}^2\right) \tag{B2}$$

and

$$\left|M_{11}\right|=\left(1-\rho_{23}^2\right)\sigma_2^2\sigma_3^2,\ \left|M_{22}\right|=\left(1-\rho_{13}^2\right)\sigma_1^2\sigma_3^2,\ \left|M_{33}\right|=\left(1-\rho_{12}^2\right)\sigma_1^2\sigma_2^2,$$

$$\left|M_{12}\right|=\left|M_{21}\right|=\sigma_1\sigma_2\sigma_3^2\left(\rho_{23}\rho_{13}-\rho_{12}\right),\ \left|M_{23}\right|=\left|M_{32}\right|=\sigma_1^2\sigma_2\sigma_3\left(\rho_{12}\rho_{13}-\rho_{23}\right),$$

$$\left|M_{13}\right|=\left|M_{31}\right|=\sigma_1\sigma_2^2\sigma_3\left(\rho_{12}\rho_{23}-\rho_{13}\right). \tag{B3}$$

The parameters σ_i^2, i = 1, 2, 3 are called variances and ρ_{ij}, i = 1, 2, 3, j = 1, 2, 3, $i \neq j$ are crosscorrelation factors. If all $\rho_{ij} = 0$, we deal with a 3-D separable Gaussian signal. The Fourier spectrum $U(\omega_1, \omega_2, \omega_3)$, $\omega_i = 2\pi f_i$ of the signal (B1) is

$$U(\omega_1, \omega_2, \omega_3)$$
$$= \exp\left[-\tfrac{1}{2}\left(\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + \omega_3^2\sigma_3^2\right)\right] \cdot \exp\left[-\left(\omega_1\omega_2\rho_{12}\sigma_1\sigma_2 + \omega_1\omega_3\rho_{13}\sigma_1\sigma_3 + \omega_2\omega_3\rho_{23}\sigma_2\sigma_3\right)\right]. \tag{B4}$$

APPENDIX C. THE SPHERE AND ITS 3-D SPECTRUM

The sphere is a spherically symmetric function

$$u(r) = \begin{cases} 1 & \text{if } r < r_1 \\ 0 & \text{if } r > r_1 \end{cases} \tag{C1}$$

Its Fourier transform is [19]

$$U(\omega) = \frac{4\pi}{\rho^3}\left[\sin(r_1\rho) - r_1\rho\cos(r_1\rho)\right] \tag{C2}$$

where $\rho = \|\omega\|$.

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[19] The derivation of (C2) is delivered to us by Prof. K. Howell.

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